4. Maxima and Minima

ABSOLUTE MAXIMUM VALUE OF A FUNCTION *A function $f(x)$ is said to have the greatest value or absolute maximum value at a point 'a' in its domain if $f(x) \leq f(a)$ for all x in the domain of $f(x)$.*

In this case, *'a'* is called the *point of maximum.*

ABSOLUTE MINIMUM VALUE OF A FUNCTION *A function $f(x)$ is said to have the smallest value or absolute minimum value at a point 'a' in its domain if $f(a) \leq f(x)$ for all x in the domain of $f(x)$.*

The point a in this case, is called the *point of minimum.*

Examples (i) Consider $f(x) = \sin x$ on the interval $[0, \pi]$.

Clearly, the greatest value of the function, i.e., the absolute maximum, is $f(\pi/2) = 1$.

Also, the smallest value of the function, i.e., the absolute minimum, is $f(0) = f(\pi) = 0$.

(ii) Consider $f(x) = x^2$ for all $x \in R$.

It is clear that the smallest value of the function, i.e., the absolute minimum, is 0.

But, we can say nothing about the greatest value of the function. In other words, we can say that the absolute maximum does not exist.

(iii) Consider $f(x) = x^3$ for all $x \in R$.

Here, we observe that as the value of x increases, the value of the function increases, and as x decreases, the value of $f(x)$ decreases.

But, the function has neither a greatest value nor a least value.

Thus, the function has neither an absolute maximum nor an absolute minimum.

(iv) Consider $f(x) = -(x-1)^2 + 5$ for all $x \in R$.

For this function, we obtain the absolute maximum when its negative part is 0, i.e, when $(x-1)^2$ is 0. So, the greatest value of the function is 5.

Clearly, it does not have a least value.

EXAMPLE 1 *Without using the derivative, find the maximum or minimum values, if any, of the function $f(x) = 4x^2 - 4x + 7$ for all $x \in R$.*

SOLUTION We have $f(x) = 4x^2 - 4x + 7 = (2x-1)^2 + 6$.

Clearly, $(2x-1)^2$ is non-negative for all $x \in R$.

The least value of $(2x-1)^2$ is 0.

So, the least value of the function is 6.

Clearly, this happens when $2x - 1 = 0$, i.e., $x = (1/2)$.

Thus, $x = (1/2)$ is a point of absolute minimum. However, $f(x)$ does not have an absolute maximum.

EXAMPLE 2 *Find the maximum or minimum values, if any, of the following functions, without using the derivatives:*

(i) $f(x) = |x + 2|$ *(ii)* $f(x) = -|x + 1| + 3$

(iii) $f(x) = |\sin(4x + 3)|$ *(iv)* $f(x) = \sin(\sin x)$

SOLUTION (i) Clearly, $|x + 2|$ is non-negative for all $x \in R$.

The least value of $|x + 2|$ is 0.

So, the minimum value of the function is 0.

Clearly, $f(x) = |x + 2|$ does not have a maximum value.

(ii) The value of $f(x) = -|x + 1| + 3$ is maximum when its negative part has a value 0. Thus, the maximum value of $f(x)$ is 3.

Clearly, there is no minimum value of $f(x)$.

(iii) The maximum value of $|\sin \theta|$ is 1, so the maximum value of $|\sin(4x + 3)|$ is 1. Also, the value of $|\sin(4x + 3)|$ is non-negative. So, the minimum value of $|\sin(4x + 3)|$ is 0.

(iv) $-1 \le \sin x \le 1$

$\Rightarrow \sin(-1) \le \sin(\sin x) \le \sin 1$

$\{\because \sin x$ is increasing on $[-1, 1]\}$

$\Rightarrow -\sin 1 \le \sin(\sin x) \le \sin 1.$

This shows that the maximum value of $f(x)$ is $\sin 1$ and the minimum value is $-\sin 1$.

Local Maxima and Local Minima

Here, our main interest lies in finding the maximum and minimum values of a function in a small interval rather than considering it for the whole of the domain. So, we shall deal with the topics of local maxima and local minima.

LOCAL MAXIMUM VALUE OF A FUNCTION *We say that c is a point of local maximum of a function $f(x)$ if there is an open interval I containing c such that $f(x) < f(c)$ for all $x \in I$.*

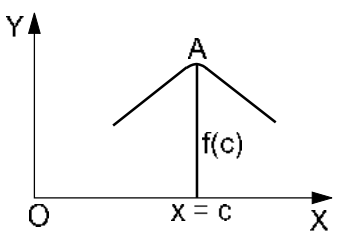
Also, in this case, $f(c)$ is called a *local maximum value* of $f(x)$.

LOCAL MINIMUM VALUE OF A FUNCTION *We say that c is a point of local minimum of a function $f(x)$ if there is an open interval I containing c such that $f(c) < f(x)$ for all $x \in I$.*

BEHAVIOUR OF $f'(x)$ AT LOCAL MAXIMA AND LOCAL MINIMA

Case I *When c is a point of local maxima*

In this case, $f(x)$ is increasing for values of x slightly less than c; it ceases to increase at $x = c$ and then decreases.



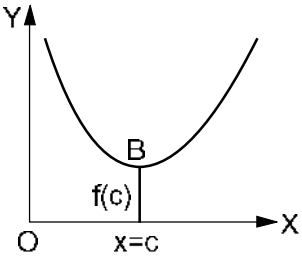
$\therefore$ for small positive values of h, we have

$f'(x) > 0$ on $[c - h, c]$

and, $f'(x) < 0$ on $[c, c + h]$.

Thus, $f'(x)$ changes sign from positive to negative as x increases through c.

Case II *When c is a point of local minima*



In this case $f(x)$ is decreasing for values of x slightly less than c; it ceases to decrease at $x = c$ and then increases for values of x slightly greater than c.

$\therefore$ for small positive values of h, we have

$$f'(x) < 0 \text{ on } [c-h, c]$$

and, $f'(x) > 0$ on $[c, c+h]$.

Thus, $f'(x)$ changes sign from negative to positive as x increases through c.

EXTREMUM VALUES *The maximum or minimum value of a function is called an extreme or extremum value of the function.*

THEOREM *If $f'(x)$ exists on an interval $[a, b]$ and $f(x)$ has a maximum or minimum value at $x = c \in]a, b[$ then $f'(c) = 0$.*

PROOF The proof is beyond the scope of this book.

STATIONARY POINTS *The points at which $f'(c) = 0$ are known as stationary points or turning points.*

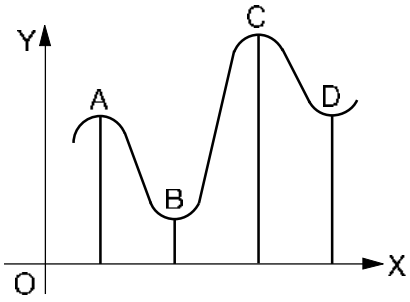
REMARKS

(i) A function can have at the most one absolute maximum and at the most one absolute minimum.

(ii) A function may have more than one local maximum and more than one local minimum.

(iii) A local minimum may be greater than a local maximum.

(iv) Local maximums and local minimums occur alternately.



In the given figure, A and C are the points of local maxima, while B and D are the points of local minima.

Clearly, the local minimum at D is greater than the local maximum at A.

Tests for Finding Local Extremum Values

We must remember that

(i) c is a point of local maximum if $f'(c) = 0$ and $f'(x)$ changes sign from positive to negative as x increases through c

(ii) c is a point of local minimum if $f'(c) = 0$ and $f'(x)$ changes sign from negative to positive as x increases through c.

FIRST-DERIVATIVE TEST

Working Rule for Finding Extremum Values

In order to find the extremum values of a given function $f(x)$ by using the first-derivative test, we proceed according to the following steps.

1. Find $f'(x)$.
2. Solve $f'(x) = 0$. Let its roots be $a, b, c,$ etc. Then, these are the candidates for maxima or minima.
3. We test the function at each one of these values. Let us take $x = c$.
4. Determine the sign of $f'(x)$ for values of x slightly less than c and for values of x slightly greater than c.

CONCLUSION (i) If $f'(x)$ changes sign from positive to negative as x increases through c then $x = c$ is a point of maximum.

(ii) If $f'(x)$ changes sign from negative to positive as x increases through c then $x = c$ is a point of minimum.

(iii) If $f'(x)$ does not change sign as x increases through c, we say that x is neither a point of maximum nor a point of minimum.
In this case, $x = c$ is called a *point of inflexion.*

Similarly, we may deal with the other roots of $f'(x) = 0$ and examine them for maxima or minima.

SOLVED EXAMPLES

EXAMPLE 1 *Find the local maxima or local minima, if any, of*

(i) $f(x) = \dfrac{1}{(x^2 + 2)}$ *(ii)* $f(x) = (x^3 - 3x)$

In each case, find the local maximum or the local minimum values, as the case may be.

SOLUTION (i) $f(x) = \dfrac{1}{(x^2 + 2)} \Rightarrow f'(x) = \dfrac{-2x}{(x^2 + 2)^2}$.

For a local maxima or minima, we have $f'(x) = 0$.

$\therefore \quad f'(x) = 0 \Rightarrow \dfrac{-2x}{(x^2 + 2)^2} = 0 \Rightarrow x = 0.$

Now, when x is slightly less than 0, i.e., when x is negative, then $f'(x) = \dfrac{-2x}{(x^2 + 2)^2}$ is positive.

When x is slightly greater than 0 then $f'(x) = \dfrac{-2x}{(x^2 + 2)^2} < 0.$

Thus, $f'(x)$ changes sign from positive to negative.

So, $x = 0$ is a point of local maximum.

And, local maximum value is $f(0) = \dfrac{1}{(0^2 + 2)} = \dfrac{1}{2}$.

(ii) $f(x) = (x^3 - 3x) \Rightarrow f'(x) = (3x^2 - 3).$

Now, for a maximum or a minimum, we have $f'(x) = 0$.

Now, $f'(x) = 0 \Rightarrow 3x^2 - 3 = 0$

$\Rightarrow 3(x - 1)(x + 1) = 0 \Rightarrow x = 1$ or $x = -1.$

Thus, $x = 1$ and $x = -1$ are the candidates for local maxima or local minima.

Consider $x = 1$.

We know that $f'(x) = 3(x^2 - 1)$.

When x is slightly less than 1 then x^2 is slightly less than 1 and therefore, $3(x^2 - 1)$ is negative.

Again, when x is slightly more than 1 then x^2 is slightly more than 1 and therefore, $3(x^2 - 1)$ is positive.

Thus, $f'(x)$ changes values from negative to positive as x increases through 1.

So, $x = 1$ is a point of local minimum.

Local minimum value $= f(1) = (1^3 - 3 \times 1) = -2$.

Consider $x = -1$.

When x is slightly less than -1 then x^2 is slightly more than 1. So, $3(x^2 - 1)$ is positive.

Again, when x is slightly more than -1 then x^2 is slightly less than 1. So, $3(x^2 - 1)$ is negative.

Thus, $f'(x)$ changes sign from positive to negative as x increases through -1.

So, $x = -1$ is a point of local maximum.

Local maximum value $= f(-1) = (-1)^3 - 3 \times (-1) = 2$.

EXAMPLE 2 *Find the local maxima or local minima of* $f(x) = x^3 - 6x^2 + 9x + 15$. *Also, find the local maximum or local minimum values as the case may be.*

SOLUTION Here, $f(x) = x^3 - 6x^2 + 9x + 15 \Rightarrow f'(x) = 3x^2 - 12x + 9$.

For a local maxima or minima, we must have $f'(x) = 0$.

Now, $f'(x) = 0 \Rightarrow 3(x^2 - 4x + 3) = 0$

$\Rightarrow 3(x - 3)(x - 1) = 0 \Rightarrow x = 3$ or $x = 1$.

Case I *When* $x = 3$

In this case, when x is slightly less than 3 then $f'(x) = 3(x - 3)(x - 1)$ is negative and when x is slightly more than 3 then $f'(x)$ is positive.

Thus, $f'(x)$ changes from negative to positive as x increases through 3.

So, $x = 3$ is a point of local minimum.

Local minimum value $= f(3) = 15$.

Case II *When* $x = 1$

In this case, when x is slightly less than 1 then $f'(x) = 3(x - 3)(x - 1)$ is positive and when x is slightly more than 1 then $f'(x)$ is negative.

Thus, $f'(x)$ changes sign from positive to negative as x increases through 1.

So, $x = 1$ is a point of local maximum.

Local maximum value $= f(1) = 19$.

EXAMPLE 3 *Show that* $(x^5 - 5x^4 + 5x^3 - 1)$ *has a local maximum when* $x = 1$, *a local minimum when* $x = 3$, *and neither when* $x = 0$.

SOLUTION Let $f(x) = x^5 - 5x^4 + 5x^3 - 1$. Then,

$$f'(x) = 5x^4 - 20x^3 + 15x^2 = 5x^2(x^2 - 4x + 3) = 5x^2(x - 3)(x - 1).$$

For a local maxima or minima, we have $f'(x) = 0$.

Now, $f'(x) = 0 \Rightarrow 5x^2(x - 3)(x - 1) = 0 \Rightarrow x = 0$ or $x = 3$ or $x = 1$.

Case I *When* $x = 1$

In this case, when x is slightly less than 1 then $f'(x) = 5x^2(x - 3)(x - 1)$ is positive and when x is slightly more than 1 then $f'(x)$ is negative.

Thus, $f'(x)$ changes sign from positive to negative.

So, $x = 1$ is a point of local maximum.

Case II *When* $x = 3$

Clearly, when x is slightly less than 3 then $f'(x) = 5x^2(x - 3)(x - 1)$ is negative and when x is slightly greater than 3 then $f'(x)$ is positive.

Thus, $f'(x)$ changes sign from negative to positive.

So, $x = 3$ is a point of local minimum.

Case III *When* $x = 0$

Clearly, when x is slightly less than 0 then $f'(x) = 5x^2(x - 3)(x - 1)$ is positive and also when x is slightly more than 0 then $f'(x)$ is positive.

Thus, $f'(x)$ does not change sign as it passes through 0.

So, it is neither a point of local maximum nor a point of local minimum, i.e., $x = 0$ is a point of inflexion.

EXAMPLE 4 *Find the points of local maxima and local minima as well as the corresponding local maximum and local minimum values for the function*

$$f(x) = (x - 1)^3(x + 1)^2.$$

SOLUTION $f(x) = (x-1)^3(x+1)^2 \Rightarrow f'(x) = 2(x-1)^3(x+1) + 3(x-1)^2(x+1)^2$

$= (x-1)^2(x+1)(5x+1).$

$\therefore \quad f'(x) = 0 \Rightarrow (x-1)^2(x+1)(5x+1) = 0$

$\Rightarrow x = 1 \text{ or } x = -1 \text{ or } x = -\frac{1}{5}.$

Consider $x = 1.$

When x is slightly less than 1 then clearly $f'(x)$ is +ve. When x is slightly greater than 1, then clearly $f'(x)$ is +ve. Thus, $f'(x)$ does not change sign as x passes through 1.

So, $x = 1$ is neither a point of maximum nor a point of minimum.

Consider $x = -1.$

When x is slightly less than -1 then $f'(x) = (+)(-)(-) = +\text{ve}.$

When x is slightly greater than -1 then $f'(x) = (+)(+)(-) = -\text{ve}.$

Thus, $f'(x)$ changes sign from +ve to –ve as x passes through $-1.$

So, $x = -1$ is a point of local maximum.

Local maximum value $= f(-1) = (-2)^3(-1+1)^2 = 0.$

Consider $x = -\frac{1}{5}.$

When x is slightly less than $-\frac{1}{5}$ then $f'(x) = (+)(+)(-) = -\text{ve}.$

When x is slightly greater than $-\frac{1}{5}$ then $f'(x) = (+)(+)(+) = +\text{ve}.$

Thus, $f'(x)$ changes sign from –ve to +ve as x passes through $-\frac{1}{5}.$

So, $x = -\frac{1}{5}$ is a point of local minimum.

Local minimum value $= f\left(-\frac{1}{5}\right) = \frac{-3456}{3125}.$

SECOND-DERIVATIVE TEST

Working Rule for Finding Extremum Values

1. Find $f'(x)$.
2. Solve $f'(x) = 0.$
   Each value of x so obtained is a candidate for maximum or minimum.
   Let $x = c$ be one of its points.
3. Find $f''(x)$ and put $x = c$ to get $f''(c)$.
   Now, if $f''(c) < 0$ then $x = c$ is a point of local maximum;
   if $f''(c) > 0$ then $x = c$ is a point of local minimum;
   if $f''(c) = 0$ then use the first-derivative test.

EXAMPLE 5 *Find all the points of local maxima and minima and the corresponding maximum and minimum values of the function*

$$f(x) = -\frac{3}{4}x^4 - 8x^3 - \frac{45}{2}x^2 + 105.$$

SOLUTION $f(x) = -\frac{3}{4}x^4 - 8x^3 - \frac{45}{2}x^2 + 105.$

$\therefore \quad f'(x) = -3x^3 - 24x^2 - 45x = -3x(x^2 + 8x + 15).$

Now, $f'(x) = 0 \Rightarrow -3x(x^2 + 8x + 15) = 0 \Rightarrow -3x(x+5)(x+3) = 0$

$\Rightarrow x = 0$ or $x = -5$ or $x = -3.$

Thus, $x = 0$, $x = -5$ and $x = -3$ are the candidates for local maxima or minima.

Moreover, $f''(x) = (-9x^2 - 48x - 45).$

Case I *When* $x = 0$

We have $f''(0) = -45 < 0.$

So, $x = 0$ is a point of local maximum.

And, local maximum value at $x = 0$ is $f(0) = 105.$

Case II *When* $x = -5$

We have $f''(-5) = -30 < 0.$

So, $x = -5$ is a point of local maximum.

And, local maximum value at $x = -5$ is $f(-5) = \frac{295}{4}.$

Case III *When* $x = -3$

We have $f''(-3) = 18 > 0.$

So, $x = -3$ is a point of local minimum.

Local minimum value at $x = -3$ is $f(-3) = \frac{231}{4}.$

EXAMPLE 6 *Find the local maxima and local minima of the functions:*

(i) $f(x) = \sin 2x$, *where* $0 < x < \pi$

(ii) $f(x) = (\sin 2x - x)$, *where* $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

SOLUTION (i) $f(x) = \sin 2x \Rightarrow f'(x) = 2\cos 2x.$

$\therefore \quad f'(x) = 0 \Rightarrow 2\cos 2x = 0,$ i.e., $\cos 2x = 0$

$\Rightarrow 2x = \frac{\pi}{2}$ or $2x = \frac{3\pi}{2} \Rightarrow x = \frac{\pi}{4}$ or $x = \frac{3\pi}{4}.$

Thus, $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$ are the candidates for local maxima or minima.

Moreover, $f'(x) = 2\cos 2x \Rightarrow f''(x) = -4\sin 2x.$

Case I *When* $x = (\pi/4)$

We have $f''\left(\frac{\pi}{4}\right) = -4\sin\frac{\pi}{2} = -4 < 0.$

$\therefore \quad x = \frac{\pi}{4}$ is a point of local maximum.

Local maximum value $= f\left(\frac{\pi}{4}\right) = 1.$

Case II *When* $x = (3\pi/4)$

We have $f''(3\pi/4) = -4\sin\frac{3\pi}{2} = 4 > 0.$

$\therefore \quad x = \frac{3\pi}{4}$ is a point of local minimum.

Local minimum value $= f\left(\frac{3\pi}{4}\right) = -1.$

(ii) $f(x) = (\sin 2x - x) \Rightarrow f'(x) = (2\cos 2x - 1)$ and $f''(x) = -4\sin 2x.$

$\therefore \quad f'(x) = 0 \Rightarrow (2\cos 2x - 1) = 0 \Rightarrow \cos 2x = \frac{1}{2}$

$\Rightarrow \quad 2x = -\frac{\pi}{3}$ or $2x = \frac{\pi}{3} \Rightarrow x = \frac{-\pi}{6}$ or $x = \frac{\pi}{6}.$

Thus, $x = \frac{-\pi}{6}$ and $x = \frac{\pi}{6}$ are the candidates for local maxima or local minima.

Case I *When* $x = -(\pi/6)$

We have $f''\left(-\frac{\pi}{6}\right) = 4\sin\frac{\pi}{3} = 2\sqrt{3} > 0.$

So, $x = \frac{-\pi}{6}$ is a point of local minimum.

The local minimum value

$$= f\left(-\frac{\pi}{6}\right) = \left(-\sin\frac{\pi}{3} - \frac{\pi}{6}\right) = -\left(\frac{\sqrt{3}}{2} + \frac{\pi}{6}\right).$$

Case II *When* $x = \frac{\pi}{6}$

We have $f''\left(\frac{\pi}{6}\right) = -4\sin\left(\frac{\pi}{3}\right) = -2\sqrt{3} < 0.$

$\therefore \quad x = \frac{\pi}{6}$ is a point of local maximum.

And, the local maximum value

$$= f\left(\frac{\pi}{6}\right) = \left(\sin\frac{\pi}{3} - \frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right).$$

EXAMPLE 7 *Find the local maxima and local minima of the functions:*

(i) $f(x) = (\sin x - \cos x)$, *where* $0 < x < \frac{\pi}{2}$

(ii) $f(x) = (2\cos x + x)$, *where* $0 < x < \pi$

SOLUTION (i) $f(x) = (\sin x - \cos x)$.

$\therefore \quad f'(x) = (\cos x + \sin x)$ and $f''(x) = (-\sin x + \cos x)$.

Now, $f'(x) = 0 \Rightarrow \cos x + \sin x = 0$

$\Rightarrow \tan x = -1 \Rightarrow x = \dfrac{3\pi}{4}$ or $\dfrac{7\pi}{4}$.

Thus, $x = \dfrac{3\pi}{4}$ and $x = \dfrac{7\pi}{4}$ are the candidates for local maxima or local minima.

Case I *When* $x = (3\pi/4)$

$$f''\left(\frac{3\pi}{4}\right) = \left(-\sin\frac{3\pi}{4} + \cos\frac{3\pi}{4}\right) = \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = \frac{-2}{\sqrt{2}}$$

$$= -\sqrt{2} < 0.$$

So, $x = (3\pi/4)$ is a point of local maximum.

Local maximum value $= f\left(\dfrac{3\pi}{4}\right) = \sqrt{2}$.

Case II *When* $x = (7\pi/4)$

$$f''\left(\frac{7\pi}{4}\right) = f''\left(-\sin\frac{7\pi}{4} + \cos\frac{7\pi}{4}\right) = \left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right)$$

$$= \sqrt{2} > 0.$$

So, $x = (7\pi/4)$ is a point of local minimum.

$$\text{Local minimum value} = f\left(\frac{7\pi}{4}\right) = \sin\frac{7\pi}{4} - \cos\frac{7\pi}{4}$$

$$= \left(-\sin\frac{\pi}{4} - \cos\frac{\pi}{4}\right) = -\sqrt{2}.$$

(ii) $f(x) = (2\cos x + x)$.

$\therefore \quad f'(x) = (-2\sin x + 1)$ and $f''(x) = -2\cos x$.

Now, $f'(x) = 0 \Rightarrow -2\sin x + 1 = 0$

$\Rightarrow \sin x = \dfrac{1}{2} \Rightarrow x = \dfrac{\pi}{6}$ or $x = \dfrac{5\pi}{6}$ in $]0, \pi[$.

Thus, $x = (\pi/6)$ and $x = (5\pi/6)$ are the candidates for local maxima or local minima.

Case I *When* $x = (\pi/6)$

We have $f''\left(\dfrac{\pi}{6}\right) = -2\cos\dfrac{\pi}{6} = \left(-2 \times \dfrac{\sqrt{3}}{2}\right) = -\sqrt{3} < 0.$

$\therefore \quad x = (\pi/6)$ is a point of local maximum.

Local maximum value $= f\left(\dfrac{\pi}{6}\right) = \left(\sqrt{3} + \dfrac{\pi}{6}\right)$.

Case II *When* $x = (5\pi/6)$

We have $f''\left(\dfrac{5\pi}{6}\right) = -2\cos\dfrac{5\pi}{6} = 2\cos\dfrac{\pi}{6} = 2 \times \dfrac{\sqrt{3}}{2} = \sqrt{3} > 0.$

So, $x = (5\pi/6)$ is a point of local minimum.

Local minimum value $= f\left(\frac{5\pi}{6}\right) = \left(\frac{5\pi}{6} - \sqrt{3}\right)$.

EXAMPLE 8 *Find the points of local maxima or local minima of the function*

$$f(x) = (\sin^4 x + \cos^4 x) \text{ in } 0 < x < \frac{\pi}{2}.$$

SOLUTION $f(x) = \sin^4 x + \cos^4 x$

$\Rightarrow f'(x) = 4\sin^3 x \cos x - 4\cos^3 x \sin x$

$= -4\sin x \cos x(\cos^2 x - \sin^2 x) = -2\sin 2x \cos 2x = -\sin 4x.$

And, $f''(x) = -4\cos 4x.$

Now, $f'(x) = 0 \Rightarrow -\sin 4x = 0 \Rightarrow 4x = \pi$, i.e., $x = \frac{\pi}{4}$.

$\therefore$ $x = (\pi/4)$ is a point of local maximum or local minimum.

Now, $f''\left(\frac{\pi}{4}\right) = -4\cos\pi = 4 > 0.$

$\therefore$ $x = (\pi/4)$ is a point of local minimum.

Local minimum value $= f\left(\frac{\pi}{4}\right) = \frac{1}{2}$.

EXAMPLE 9 *Find the local maxima and local minima, and the corresponding local maximum and local minimum values of the following functions:*

(i) $f(x) = x\sqrt{1-x}$, *where* $x > 0$

(ii) $f(x) = \frac{x}{(x-1)(x-4)}$, *where* $1 < x < 4$

SOLUTION (i) $f(x) = x\sqrt{1-x}.$

$\therefore$ $f'(x) = \frac{(2-3x)}{2\sqrt{1-x}}$ and $f''(x) = \frac{(3x-4)}{4(1-x)^{3/2}}$.

Now, $f'(x) = 0 \Rightarrow (2-3x) = 0 \Rightarrow x = \frac{2}{3}$;

and, $f''\left(\frac{2}{3}\right) = \frac{\left(3\times\frac{2}{3} - 4\right)}{4\left(1-\frac{2}{3}\right)^{3/2}} = \frac{-3^{3/2}}{2} < 0.$

$\therefore$ $x = \frac{2}{3}$ is a point of local maximum.

Local maximum value $= f\left(\frac{2}{3}\right) = \frac{2}{3\sqrt{3}}$.

(ii) $f(x) = \frac{x}{(x-1)(x-4)} = \frac{x}{(x^2-5x+4)}$.

$\therefore$ $f'(x) = \frac{(4-x^2)}{(x^2-5x+4)^2}$.

And,

$$f''(x) = \frac{(x^2-5x+4)^2(-2x) - (4-x^2)2(x^2-5x+4)(2x-5)}{(x^2-5x+4)^4}.$$

$$f'(x) = 0 \Rightarrow (4 - x^2) = 0 \Rightarrow x = 2 \quad [\because 1 < x < 4].$$

$$f''(2) = \frac{-16}{16} = -1 < 0.$$

$\therefore$ $x = 2$ is a point of local maximum.

Local maximum value $= f(2) = -1$.

EXAMPLE 10 *Prove that the maximum value of* $\left(\frac{1}{x}\right)^x$ *is* $e^{1/e}$.

SOLUTION Let $y = \left(\frac{1}{x}\right)^x = x^{-x}$. Then, $\log y = -x \log x$.

$$\therefore \quad \frac{1}{y} \cdot \frac{dy}{dx} = -(1 + \log x) \quad \text{or} \quad \frac{dy}{dx} = -y(1 + \log x).$$

$$\text{And, } \frac{d^2y}{dx^2} = -y \cdot \frac{1}{x} - (1 + \log x) \cdot \frac{dy}{dx} = -\left(\frac{1}{x}\right)^{x+1} - (1 + \log x) \cdot \frac{dy}{dx}.$$

$$\text{Now, } \frac{dy}{dx} = 0 \Rightarrow 1 + \log x = 0$$

$$\Rightarrow \log x = -1 = -\log e = \log (1/e) \Rightarrow x = (1/e).$$

$$\text{Also, } \frac{d^2y}{dx^2} \text{ at } x = (1/e) \text{ is } -e^{1+(1/e)} < 0 \quad [\because \frac{dy}{dx} = 0\,].$$

So, $x = (1/e)$ is a point of local maximum.

Local maximum value $= \{\text{value of } y \text{ when } x = (1/e)\} = e^{1/e}$.

EXAMPLE 11 *Find the point on the parabola* $y^2 = 2x$ *which is closest to the point* (1, 4).

SOLUTION Let $A(x, y)$ be the required point which is closest to the point $B(1, 4)$. Then, the distance AB should be minimum. And, therefore AB^2 should be minimum.

$$\text{Now, } AB^2 = (x-1)^2 + (y-4)^2 = \left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2 = \frac{(y^4 - 32y + 68)}{4}.$$

$$\text{Let } f(y) = \frac{y^4 - 32y + 68}{4}.$$

$$\text{Then, } f'(y) = \frac{4y^3 - 32}{4} = (y^3 - 8). \quad \text{And, } f''(y) = 3y^2.$$

Now, $f'(y) = 0 \Rightarrow y^3 - 8 = 0 \Rightarrow y = 2$. Also, $f''(2) = 3 \times 4 = 12 > 0$.

So, $y = 2$ is a point of minimum.

$$\text{Now, } y = 2 \Rightarrow x = \frac{y^2}{2} = \frac{4}{2} = 2. \quad \text{So, the required point is (2, 2).}$$

EXAMPLE 12 *Show that none of the following functions has a maxima or minima.*
(i) e^x *(ii)* $\log x$ *(iii)* $x^3 + x^2 + x + 1$

SOLUTION (i) $f(x) = e^x \Rightarrow f'(x) = e^x$.

For a maxima or a minima, we must have $f'(x) = 0$.

But, $f'(x) = 0 \Rightarrow e^x = 0$, which is not possible.

So, $f(x) = e^x$ does not have a maxima or a minima.

(ii) $f(x) = \log x \Rightarrow f'(x) = \frac{1}{x}$.

For a maxima or a minima, we must have $f'(x) = 0$.

But, $f'(x) = 0 \Rightarrow \frac{1}{x} = 0$, which is not possible for any value of x.

$\therefore$ $f(x) = \log x$ does not have a maxima or a minima.

(iii) $f(x) = x^3 + x^2 + x + 1 \Rightarrow f'(x) = 3x^2 + 2x + 1$.

For a maxima or a minima, we must have $f'(x) = 0$.

Now, $f'(x) = 0 \Rightarrow 3x^2 + 2x + 1 = 0$

$\Rightarrow x = \frac{-2 \pm \sqrt{4 - 12}}{6}$, which are imaginary.

Thus, $f'(x) \neq 0$ for any real value of x.

Hence, $f(x)$ does not have a maxima or a minima.

EXAMPLE 13 *Show that* $\sin^p\theta \cos^q\theta$ *attains a maximum when* $\theta = \tan^{-1}\sqrt{\frac{p}{q}}$.

SOLUTION Let $y = \sin^p 0 \cos^q 0$.

Then, $\frac{dy}{d\theta} = p \sin^{p-1}\theta \cos^{q+1}\theta - q \cos^{q-1}\theta \sin^{p+1}\theta$

$= (\sin^{p-1}\theta \cos^{q-1}\theta)(p \cos^2\theta - q \sin^2\theta)$.

Now, for maxima or minima, we have $\frac{dy}{d\theta} = 0$.

But $\frac{dy}{d\theta} = 0 \Rightarrow \sin^{p-1}0 = 0$ or $\cos^{q-1}0 = 0$ or $p\cos^2 0 - q\sin^2 0 = 0$

$\Rightarrow 0 = 0$ or $0 = \frac{\pi}{2}$ or $0 = \tan^{-1}\sqrt{\frac{p}{q}}$.

Moreover, we may write

$$\frac{dy}{d0} = \frac{y(p\cos^2 0 - q\sin^2 0)}{\sin 0 \cos 0} = y(p \cot 0 - q \tan 0).$$

$$\therefore \quad \frac{d^2y}{d0^2} = y(-p \operatorname{cosec}^2 0 - q \sec^2 0) + (p \cot 0 - q \tan 0) \cdot \frac{dy}{d\theta}.$$

Thus, at $\theta = \tan^{-1}\sqrt{\frac{p}{q}}$, we have $\frac{dy}{d\theta} = 0$ and therefore,

$$\left[\frac{d^2y}{d0^2} \text{ at } \theta = \tan^{-1}\sqrt{\frac{p}{q}}\right] = \sin^p\theta \cos^q\theta(-p \operatorname{cosec}^2\theta - q\sec^2\theta) < 0.$$

Hence, y is maximum when $\theta = \tan^{-1}\sqrt{\frac{p}{q}}$.

MAXIMA AND MINIMA ON A CLOSED INTERVAL Let $f(x)$ be a given function defined on $[a, b]$. Let the local minimum value of $f(x)$ be m, and let the local maximum value of $f(x)$ be M.

Then, *minimum value of* $f(x)$ *on* $[a, b]$ *is the smallest of* m, $f(a)$ *and* $f(b)$.

The maximum value of $f(x)$ *on* $[a, b]$ *is the greatest of* M, $f(a)$ *and* $f(b)$.

EXAMPLE 14 *Find the maximum and minimum values of*

$$(3x^4 - 8x^3 + 12x^2 - 48x + 25) \quad \textit{on} \quad [0, 3].$$

SOLUTION Let $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25.$

Then, $f'(x) = 12x^3 - 24x^2 + 24x - 48$

and $f''(x) = 36x^2 - 48x + 24.$

Now, $f'(x) = 0 \Rightarrow 12(x^3 - 2x^2 + 2x - 4) = 0$

$\Rightarrow 12(x-2)(x^2+2) = 0$

$\Rightarrow x = 2$ [neglecting the imaginary values].

And, $f''(2) = 72 > 0.$

So, $x = 2$ is a point of local minima.

Now, $f(2) = -39;\ f(0) = 25$ and $f(3) = 16.$

$\therefore$ minimum value of $f(x)$ on $[0, 3]$ is -39 at $x = 2.$

Maximum value of $f(x)$ on $[0, 3]$ is 25 at $x = 0.$

EXAMPLE 15 *Find the maximum and minimum values of* $(x + \sin 2x)$ *on* $[0, 2\pi]$.

SOLUTION Let $f(x) = (x + \sin 2x).$

Then, $f'(x) = (1 + 2\cos 2x)$ and $f''(x) = -4\sin 2x.$

Now, $f'(x) = 0 \Rightarrow \cos 2x = -\frac{1}{2}$, where $x \in [0, 2\pi]$

$\Rightarrow 2x = \frac{2\pi}{3}$ or $2x = \frac{4\pi}{3} \Rightarrow x = (\pi/3)$ or $x = (2\pi/3).$

$$f''\left(\frac{\pi}{3}\right) = -4\sin\left(\frac{2\pi}{3}\right) = -4\sin\frac{\pi}{3} = -2\sqrt{3} < 0.$$

$\therefore$ $x = (\pi/3)$ is a point of local maxima.

Now, $f\left(\frac{\pi}{3}\right) = \frac{2\pi + 3\sqrt{3}}{6}$, $f(0) = 0$ and $f(2\pi) = 2\pi.$

$\therefore$ maximum value of $f(x)$ is 2π at $x = 2\pi.$

Again, we consider the point $x = (2\pi/3).$

$$f''\left(\frac{2\pi}{3}\right) = -4\sin\left(2 \times \frac{2\pi}{3}\right) = -4\sin\frac{4\pi}{3} = 4\sin\frac{\pi}{3} = 2\sqrt{3} > 0.$$

So, $x = \frac{2\pi}{3}$ is a point of local minima.

Now, $f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + \sin\frac{4\pi}{3} = \frac{2\pi}{3} - \sin\frac{\pi}{3} = \frac{4\pi - 3\sqrt{3}}{6};$

$f(0) = 0$ and $f(2\pi) = 2\pi + \sin 4\pi = 2\pi.$

$\therefore$ the minimum value of $f(x)$ is 0 at $x = 0.$

EXAMPLE 16 *Show that* $\sin x(1 + \cos x),\ x \in [0, \pi]$ *is maximum at* $x = (\pi/3)$.

SOLUTION Let $f(x) = \sin x(1 + \cos x).$

Then, $f'(x) = -\sin^2 x + \cos x(1 + \cos x)$

$= 2\cos^2 x + \cos x - 1 = (2\cos x - 1)(\cos x + 1).$

And, $f''(x) = (-4\cos x \sin x - \sin x) = -\sin x(1 + 4\cos x).$

Now, $f'(x) = 0 \Rightarrow (2\cos x - 1)(\cos x + 1) = 0$

$\Rightarrow \cos x = \frac{1}{2}$ or $\cos x = -1 \Rightarrow x = \frac{\pi}{3}$ or $x = \pi.$

But, $f''\left(\frac{\pi}{3}\right) = -\sin\frac{\pi}{3}\left(1 + 4\cos\frac{\pi}{3}\right) = \frac{-3\sqrt{3}}{2} < 0.$

$\therefore$ $x = (\pi/3)$ is a point of local maximum.

Now, $f(0) = 0, f(\pi) = 0$ and $f\left(\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{4}.$

$\therefore$ the maximum value of $f(x)$ is $\frac{3\sqrt{3}}{4}$ at $x = \frac{\pi}{3}.$

EXERCISE 11E

Find the maximum or minimum values, if any, without using derivatives, of the functions:

1. $(5x - 1)^2 + 4$
2. $-(x - 3)^2 + 9$
3. $-|x + 4| + 6$
4. $\sin 2x + 5$
5. $|\sin 4x + 3|$

Find the points of local maxima or local minima and the corresponding local maximum and minimum values of each of the following functions:

6. $f(x) = (x - 3)^4$
7. $f(x) = x^2$
8. $f(x) = 2x^3 - 21x^2 + 36x - 20$
9. $f(x) = x^3 - 6x^2 + 9x + 15$
10. $f(x) = x^4 - 62x^2 + 120x + 9$
11. $f(x) = -x^3 + 12x^2 - 5$
12. $f(x) = (x - 1)(x + 2)^2$
13. $f(x) = -(x - 1)^3(x + 1)^2$
14. $f(x) = \frac{x}{2} + \frac{2}{x}, x > 0$
15. Find the maximum and minimum values of $2x^3 - 24x + 107$ on the interval $[-3, 3]$.
16. Find the maximum and minimum values of $3x^4 - 8x^3 + 12x^2 - 48x + 1$ on the interval $[1, 4]$.
17. Find the maximum and minimum values of
$$f(x) = \left(\sin x + \frac{1}{2}\cos x\right) \text{ in } 0 \le x \le \frac{\pi}{2}.$$
18. Show that the maximum value of $x^{1/x}$ is $e^{1/e}$.
19. Show that $\left(x + \frac{1}{x}\right)$ has a maximum and minimum, but the maximum value is less than the minimum value.

20. Find the maximum profit that a company can make, if the profit function is given by $p(x) = 41 + 24x - 18x^2$.

21. An enemy jet is flying along the curve $y = (x^2 + 2)$. A soldier is placed at the point $(3, 2)$. Find the nearest point between the soldier and the jet.

22. Find the maximum and minimum values of
$$f(x) = (-x + 2\sin x) \text{ on } [0, 2\pi].$$

ANSWERS (EXERCISE 11E)

1. min. value $= 4$ **2.** max. value $= 9$ **3.** max. value $= 6$

4. max. value $= 4$, min. value $= 6$ **5.** max. value $= 4$, min. value $= 2$

6. local max. value is 0 at $x = 3$ **7.** local min. value is 0 at $x = 0$

8. local max. value is -3 at $x = 1$ and local min. value is -128 at $x = 6$

9. local max. value is 19 at $x = 1$ and local min. value is 15 at $x = 3$

10. local max. value is 68 at $x = 1$ and local min. values are -1647 at $x = -6$ and -316 at $x = 5$

11. local max. value is 251 at $x = 8$ and local min. value is -5 at $x = 0$

12. local max. value is 0 at $x = -2$ and local min. value is -4 at $x = 0$

13. local max. value is 0 at each of the points $x = 1$ and $x = -1$ and local min. value is $\dfrac{-3456}{3125}$ at $x = -\dfrac{1}{5}$

14. local min. value is 2 at $x = 2$

15. max. value is 139 at $x = -2$ and min. value is 89 at $x = 3$

16. max. value is 257 at $x = 4$ and min. value is -63 at $x = 2$

17. max. value is $\dfrac{3}{4}$ at $x = \dfrac{\pi}{6}$ and min. value is $\dfrac{1}{2}$ at $x = \dfrac{\pi}{2}$

20. 49 **21.** $(1, 3)$

22. max. value is $\left(-\dfrac{\pi}{3} + \sqrt{3}\right)$ at $x = \dfrac{\pi}{3}$ and min. value is $\left(\dfrac{5\pi}{3} + \sqrt{3}\right)$ at $x = \dfrac{5\pi}{3}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 11E)

4. $-1 \le \sin 2x \le 1$.

5. $-1 \le \sin 4x \le 1 \Rightarrow (-1 + 3) \le (\sin 4x + 3) \le (1 + 3)$.

6. Use the first-derivative test.

21. Find a point $(x, x^2 + 2)$ nearest to $(3, 2)$.

SOME MORE SOLVED EXAMPLES

EXAMPLE 1 *Amongst all pairs of positive numbers with sum* 24, *find those whose product is maximum.*

SOLUTION Let the numbers be x and $(24-x)$.

Let $P = x(24-x) = (24x - x^2)$.

Then, $\dfrac{dP}{dx} = (24-2x)$ and $\dfrac{d^2P}{dx^2} = -2$.

Now, $\dfrac{dP}{dx} = 0 \Rightarrow (24-2x) = 0 \Rightarrow x = 12$.

Thus, $\left\{\dfrac{d^2P}{dx^2}\right\}_{x=12} = -2 < 0$.

$\therefore$ $x = 12$ is a point of maximum.

Hence, the required numbers are 12 and 12.

EXAMPLE 2 *Amongst all pairs of positive numbers with product* 256, *find those whose sum is the least.*

SOLUTION Let the required numbers be x and $\dfrac{256}{x}$.

Let $S = \left(x + \dfrac{256}{x}\right)$. Then, $\dfrac{dS}{dx} = \left(1 - \dfrac{256}{x^2}\right)$ and $\dfrac{d^2S}{dx^2} = \dfrac{512}{x^3}$.

For a maxima or minima, we have $\dfrac{dS}{dx} = 0$.

Now, $\dfrac{dS}{dx} = 0 \Rightarrow \left(1 - \dfrac{256}{x^2}\right) = 0 \Rightarrow x^2 = 256$ or $x = 16$.

Also, $\left[\dfrac{d^2S}{dx^2}\right]_{x=16} = \left(\dfrac{512}{16 \times 16 \times 16}\right) = \dfrac{1}{8} > 0$.

So, $x = 16$ is a point of minimum.

Hence, the required numbers are 16 and 16.

EXAMPLE 3 *Find two positive numbers x and y such that $(x+y) = 60$ and xy^3 is maximum.*

SOLUTION Let $(x+y) = 60$ and let $P = xy^3$.

Then, $P = (60-y)y^3$ $\quad [\because\ x = (60-y)]$.

$\therefore$ $\dfrac{dP}{dy} = 3y^2(60-y) + y^3(-1) = (180y^2 - 4y^3) = 4y^2(45-y)$

and, $\dfrac{d^2P}{dy^2} = (360y - 12y^2) = 12y(30-y)$.

Now, $\dfrac{dP}{dy} = 0 \Rightarrow 4y^2(45-y) = 0 \Rightarrow y = 0$ or $y = 45$.

Neglecting $y = 0$, we are left with $y = 45$.

Now, $\left[\frac{d^2P}{dy^2}\right]_{y=45} = (12 \times 45)(30 - 45) = -8100 < 0.$

So, $y = 45$ is a point of maximum.

Hence, the numbers are 45 and 15.

EXAMPLE 4 *Find two numbers whose sum is* 16 *and the sum of whose cubes is minimum.*

SOLUTION Let the numbers be x and $(16 - x)$.

Let $S = x^3 + (16 - x)^3$.

Then, $\frac{dS}{dx} = 3x^2 - 3(16 - x)^2$ and $\frac{d^2S}{dx^2} = 6x + 6(16 - x) = 96.$

Now, $\frac{dS}{dx} = 0 \Rightarrow 3x^2 - 3(16 - x)^2 = 0 \Rightarrow x = 8.$

And, $\left[\frac{d^2S}{dx^2}\right]_{x=8} = 96 > 0.$

$\therefore$ $x = 8$ is a point of minimum.

Hence, the required numbers are 8 and 8.

EXAMPLE 5 *Show that of all the rectangles with a given perimeter, the square has the largest area.*

SOLUTION Let a be the fixed perimeter.

Consider a rectangle with sides x and y and perimeter a.

Let A be the area of the rectangle.

Then, $2x + 2y = a$ … (i)

And, $A = xy = x\left(\frac{a - 2x}{2}\right)$ [using (i)].

$\therefore$ $\frac{dA}{dx} = \left(\frac{1}{2}a - 2x\right) = \frac{1}{2}(a - 4x)$ and $\frac{d^2A}{dx^2} = -2.$

Now, $\frac{dA}{dx} = 0 \Rightarrow \frac{1}{2}(a - 4x) = 0 \Rightarrow x = \frac{a}{4}.$

Also, $\left[\frac{d^2A}{dx^2}\right]_{x=(a/4)} = -2 < 0.$

$\therefore$ $x = (a/4)$ is a point of maximum.

Also, from (i), we have $y = \frac{1}{2}(a - 2x) = \frac{1}{2}\left(a - \frac{a}{2}\right) = \frac{1}{4}a$, when $x = \frac{a}{4}.$

Thus, $x = y$ [each $= (a/4)$].

Hence, the rectangle is a square.

EXAMPLE 6 *Show that of all the rectangles of a given area, the square has the smallest perimeter.* **[CBSE 2011]**

SOLUTION Let A be the fixed area.

Consider a rectangle with sides x and y, and area A.

Let P be the perimeter of this rectangle.

Then, $A = xy \Rightarrow y = \frac{A}{x}$ … (i)

And, $P = 2x + 2y = 2x + \frac{2A}{x}$ [using (i)].

Now, $P = \left(2x + \frac{2A}{x}\right) \Rightarrow \frac{dP}{dx} = \left(2 - \frac{2A}{x^2}\right)$ and $\frac{d^2P}{dx^2} = \frac{4A}{x^3}$.

Now, $\frac{dP}{dx} = 0 \Rightarrow 2 - \frac{2A}{x^2} = 0 \Rightarrow x = \sqrt{A}$

and, $\left[\frac{d^2P}{dx^2}\right]_{x=\sqrt{A}} = \frac{4A}{A^{3/2}} = \frac{4}{\sqrt{A}} > 0.$

So, $x = \sqrt{A}$ is a point of minimum.

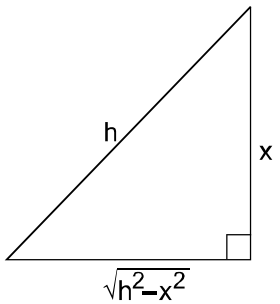
Moreover, $x = \sqrt{A} \Rightarrow y = \frac{A}{x} = \frac{A}{\sqrt{A}} = \sqrt{A} = x.$

So, when the perimeter is smallest, the rectangle is a square.

EXAMPLE 7 *Prove that the area of a right-angled triangle of a given hypotenuse is maximum when the triangle is isosceles.* **[CBSE 2012C]**

SOLUTION Let h be the hypotenuse of the triangle and x be its altitude.

Then, the base of the triangle $= \sqrt{h^2 - x^2}$.



Now, $A = \frac{1}{2}x\sqrt{h^2 - x^2}$

$\therefore \quad \frac{dA}{dx} = \frac{1}{2}\left\{x \cdot \frac{1}{2}(h^2 - x^2)^{-1/2}(-2x) + \sqrt{h^2 - x^2}\right\}$

$= \frac{1}{2}\left(\frac{h^2 - 2x^2}{\sqrt{h^2 - x^2}}\right)$

and, $\frac{d^2A}{dx^2} = \frac{(\sqrt{h^2 - x^2})(-4x) - (h^2 - 2x^2) \cdot \frac{1}{2}(h^2 - x^2)^{-1/2}(-2x)}{2(h^2 - x^2)}$

$= \frac{(2x^3 - 3xh^2)}{2(h^2 - x^2)^{3/2}}.$

Now, $\frac{dA}{dx} = 0 \Rightarrow (h^2 - 2x^2) = 0 \Rightarrow x = \frac{h}{\sqrt{2}}.$

$\therefore \quad \left[\frac{d^2A}{dx^2}\right]_{x=\frac{h}{\sqrt{2}}} = -2 < 0.$

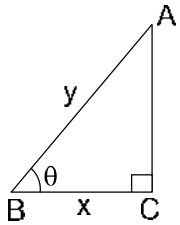
Thus, A is maximum at $x = (h/\sqrt{2})$.

$$\therefore \quad \text{base} = \sqrt{h^2 - x^2} = \sqrt{\left(h^2 - \frac{h^2}{2}\right)} = \frac{h}{\sqrt{2}} = \text{altitude}.$$

Hence, the triangle is isosceles.

EXAMPLE 8 *If the sum of the lengths of the hypotenuse and a side of a right-angled triangle is given, show that the area of the triangle is maximum when the angle between them is* $(\pi/3)$. **[CBSE 2009]**

SOLUTION Let us consider a right-angled triangle with base $= x$ and hypotenuse $= y$. Let $x + y = k$, where k is a constant. Let θ be the angle between the base and the hypotenuse. Let A be the area of the triangle. Then,



$$A = \frac{1}{2} \times BC \times AC = \frac{1}{2} x\sqrt{y^2 - x^2}.$$

$$\therefore \quad A^2 = \frac{x^2(y^2 - x^2)}{4} = \frac{x^2[(k-x)^2 - x^2]}{4} \quad [\because \ y = (k - x)]$$

$$\text{or} \quad A^2 = \frac{k^2x^2 - 2kx^3}{4} \qquad \ldots \text{(i)}$$

On differentiating (i), we get $\quad 2A \cdot \dfrac{dA}{dx} = \dfrac{2k^2x - 6kx^2}{4} \qquad \ldots$ (ii)

$$\text{or} \quad \frac{dA}{dx} = \frac{k^2x - 3kx^2}{4A}.$$

Now, $\quad \dfrac{dA}{dx} = 0 \ \Rightarrow \ (k^2x - 3kx^2) = 0 \ \Rightarrow \ x = \dfrac{k}{3}$ [neglecting $x = 0$].

Now, differentiating (ii), we get

$$2\left(\frac{dA}{dx}\right)^2 + 2A \cdot \frac{d^2A}{dx^2} = \frac{2k^2 - 12kx}{4} \qquad \ldots \text{(iii)}$$

Putting $\dfrac{dA}{dx} = 0$ and $x = \dfrac{k}{3}$ in (iii), we get $\dfrac{d^2A}{dx^2} = \dfrac{-k^2}{4A} < 0$.

Thus, A is maximum when $x = (k/3)$.

Now, $\quad x = \dfrac{k}{3} \ \Rightarrow \ y = \left(k - \dfrac{k}{3}\right) = \dfrac{2k}{3}.$

$$\therefore \quad \frac{x}{y} = \cos\theta \ \Rightarrow \ \cos\theta = \frac{(k/3)}{(2k/3)} = \frac{1}{2} \ \Rightarrow \ \theta = \frac{\pi}{3}.$$

Hence, the area is maximum when $\theta = \dfrac{\pi}{3}$.

EXAMPLE 9 *Two sides of a triangle are given. Find the angle between them such that the area is maximum.*

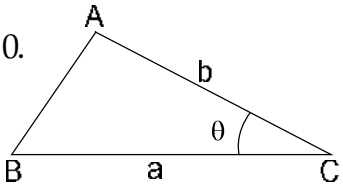
SOLUTION Let a and b be the lengths of the given sides and let θ be the angle between them. Let A be the area of the triangle.

Then, $\quad A = \dfrac{1}{2}ab \sin\theta.$

$\therefore \quad \dfrac{dA}{d\theta} = \dfrac{1}{2}ab\cos\theta$ and $\dfrac{d^2A}{d\theta^2} = -\dfrac{1}{2}ab\sin\theta$.

Now, $\dfrac{dA}{d\theta} = 0 \Rightarrow \dfrac{1}{2}ab\cos\theta = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \dfrac{\pi}{2}$.

And, $\left[\dfrac{d^2A}{d\theta^2}\right]_{\theta=(\pi/2)} = -\dfrac{1}{2}ab\sin\dfrac{\pi}{2} = -\dfrac{1}{2}ab < 0$.



$\therefore \quad \theta = \dfrac{\pi}{2}$ is a point of maximum.

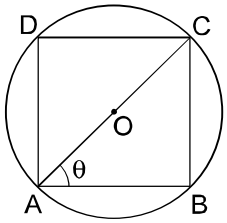
Hence, the area of the triangle is maximum when the angle between the given sides is $(\pi/2)$.

EXAMPLE 10 *Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.* **[CBSE 2008, '13C]**

SOLUTION Let $ABCD$ be a rectangle inscribed in a given circle with centre O and radius r.

Let $\angle CAB = \theta$.

Then, $AC = 2r$, $AB = 2r\cos\theta$ and $BC = 2r\sin\theta$.



Let A be the area of rectangle $ABCD$.

Then, $A = AB \times BC = 4r^2\sin\theta\cos\theta = 2r^2\sin 2\theta$.

Thus, $A = 2r^2\sin 2\theta$, where r is constant.

$\therefore \quad \dfrac{dA}{d\theta} = 4r^2\cos 2\theta$ and $\dfrac{d^2A}{d\theta^2} = -8r^2\sin 2\theta$.

Now, $\dfrac{dA}{d\theta} = 0 \Rightarrow 4r^2\cos 2\theta = 0 \Rightarrow \cos 2\theta = 0 \Rightarrow 2\theta = \dfrac{\pi}{2}$, i.e., $\theta = \dfrac{\pi}{4}$.

And, $\left[\dfrac{d^2A}{d\theta^2}\right]_{\theta=(\pi/4)} = -8r^2\sin\dfrac{\pi}{2} = -8r^2 < 0$.

$\therefore \quad \theta = (\pi/4)$ is a point of maximum.

Thus, area is maximum when $\theta = (\pi/4)$.

In this case, $AB = 2r\cos\dfrac{\pi}{4} = r\sqrt{2}$

and, $BC = 2r\sin\dfrac{\pi}{4} = r\sqrt{2}$.

Thus, $AB = BC$ and therefore, $ABCD$ is a square.

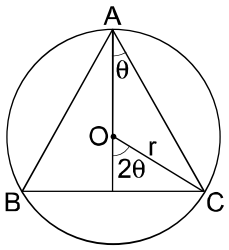
EXAMPLE 11 *Show that the triangle of maximum area that can be inscribed in a given circle is an equilateral triangle.*

SOLUTION Let ABC be a triangle inscribed in a given circle with centre O and radius r.

For maximum area, the vertex A should be at a maximum distance from the base BC.

Therefore, A must lie on the diameter, perpendicular to BC. Thus, $AD \perp BC$.

So, triangle ABC must be isosceles.



Let $\angle CAD = \theta$.

Now, $BC = 2CD = 2OC \sin 2\theta = 2r \sin 2\theta$

and, $AD = (OA + OD) = (r + r\cos 2\theta)$.

Let A be the area of the triangle.

Then, $A = \frac{1}{2} BC \times AD = r^2 \sin 2\theta (1 + \cos 2\theta)$.

$$\therefore \quad \frac{dA}{d\theta} = r^2[\sin 2\theta(-2\sin 2\theta) + (1 + \cos 2\theta) \cdot 2\cos 2\theta]$$

$$= r^2[2(\cos^2 2\theta - \sin^2 2\theta) + 2\cos 2\theta] = 2r^2[\cos 4\theta + \cos 2\theta].$$

And, $\frac{d^2A}{d\theta^2} = 2r^2[-4\sin 4\theta - 2\sin 2\theta] = -4r^2(2\sin 4\theta + \sin 2\theta)$.

Now, $\frac{dA}{d\theta} = 0 \Rightarrow \cos 4\theta + \cos 2\theta = 0$

$$\Rightarrow \cos 4\theta = -\cos 2\theta = \cos(\pi - 2\theta)$$

$$\Rightarrow 4\theta = \pi - 2\theta \Rightarrow \theta = \frac{\pi}{6}.$$

And, $\left[\frac{d^2A}{d\theta^2}\right]_{\theta = (\pi/6)} = -4r^2\left(2\sin\frac{2\pi}{3} + \sin\frac{\pi}{3}\right) = -6r^2\sqrt{3} < 0.$

$\therefore \quad \theta = \frac{\pi}{6}$ is a point of maximum.

So, in this case, each angle of the triangle is $(\pi/3)$.

Hence, ABC is an equilateral triangle.

EXAMPLE 12 *The combined resistance R of two resistors R_1 and R_2 where $R_1, R_2 > 0$ is given by* $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$.

If $R_1 + R_2 = C$ (constant), show that the maximum resistance R is obtained by choosing $R_1 = R_2$.

SOLUTION We have $R = \frac{R_1 R_2}{(R_1 + R_2)}$ and $R_1 + R_2 = C$.

$$\therefore \quad R = \frac{R_1(C - R_1)}{C} \qquad [\because \ R_2 = (C - R_1)].$$

So, $\frac{dR}{dR_1} = \frac{C - 2R_1}{C}$ and $\frac{d^2R}{dR_1^2} = -\frac{2}{C}$.

For a maxima or minima, we have $\frac{dR}{dR_1} = 0$.

Now, $\frac{dR}{dR_1} = 0 \Rightarrow \frac{C - 2R_1}{C} = 0$ or $R_1 = \frac{C}{2}$.

And, $\left[\frac{d^2R}{dR_1^2}\right]_{R_1=(C/2)} = \frac{-2}{C} < 0.$

Thus, $R_1 = \frac{C}{2}$ is a point of maximum.

When, $R_1 = \frac{C}{2}$, we have $R_2 = \left(C - \frac{C}{2}\right) = \frac{C}{2}$. Hence, $R_1 = R_2$.

EXAMPLE 13 *A beam of length l is supported at one end. If W is the uniform load per unit length, the bending moment M at a distance x from the end is given by* $M = \left(\frac{lx}{2} - \frac{Wx^2}{2}\right)$. *Find the point on the beam at which the bending moment has the maximum value.*

SOLUTION $M = \left(\frac{lx}{2} - \frac{Wx^2}{2}\right) \Rightarrow \frac{dM}{dx} = \left(\frac{l}{2} - Wx\right)$ and $\frac{d^2M}{dx^2} = -W.$

For a maxima or minima, we have $\frac{dM}{dx} = 0.$

Now, $\frac{dM}{dx} = 0 \Rightarrow \frac{l}{2} - Wx = 0 \Rightarrow x = \frac{l}{2W}$.

Also, $\frac{d^2M}{dx^2} = -W < 0$ for all values of x.

$\therefore$ $x = \frac{l}{2W}$ is a point of maxima.

So, the required point is at a distance of $(l/2W)$ from the supporting end.

EXAMPLE 14 *A wire of length* 25 m *is to be cut into two pieces. One of the wires is to be made into a square and the other into a circle. What should be the lengths of the two pieces so that the combined area of the square and the circle is minimum?* **[CBSE 1996]**

SOLUTION Let x metres and $(25 - x)$ metres be the required lengths.

Let a be the side of the square formed and r be the radius of the circle formed. Then,

$$4a = x \text{ and } 2\pi r = 25 - x \quad \text{or} \quad a = \frac{x}{4} \text{ and } r = \frac{25 - x}{2\pi}.$$

$\therefore$ area of the square $= \left(\frac{x^2}{16}\right)$ sq metres.

Area of the circle $= \pi\left(\frac{25 - x}{2\pi}\right)^2 = \frac{(25 - x)^2}{4\pi}$.

Now, the combined area $A = \frac{x^2}{16} + \frac{(25-x)^2}{4\pi}$.

$\therefore \quad \frac{dA}{dx} = \frac{x}{8} - \frac{(25-x)}{2\pi} = \frac{(\pi+4)x - 100}{8\pi}$, and $\frac{d^2A}{dx^2} = \frac{(\pi+4)}{8\pi}$.

For a maxima or minima, we have $\frac{dA}{dx} = 0$.

Now, $\frac{dA}{dx} = 0 \Rightarrow (\pi+4)x - 100 = 0 \Rightarrow x = \frac{100}{(\pi+4)}$.

And, $\frac{d^2A}{dx^2} = \frac{(\pi+4)}{8\pi} > 0$ for all values of x.

$\therefore \quad x = \frac{100}{(\pi+4)}$ is a point of minimum.

$\therefore$ lengths of the pieces are $\left(\frac{100}{\pi+4}\right)$ metres and $\left(\frac{25\pi}{\pi+4}\right)$ metres.

Some Examples Related to Volume and Area of Solids

EXAMPLE 15 *Show that a cylinder of a given volume which is open at the top has minimum total surface area, provided its height is equal to the radius of its base.* **[CBSE 2009C, '14]**

SOLUTION Let r be the radius and h the height of the cylinder of given volume V.

Then, $V = \pi r^2 h \Rightarrow h = \frac{V}{\pi r^2}$ … (i)

Let the total surface area be S. Then,

$$S = \pi r^2 + 2\pi r h = \left(\pi r^2 + 2\pi r \cdot \frac{V}{\pi r^2}\right) \qquad \text{[using (i)]}.$$

$\therefore \quad S = \left(\pi r^2 + \frac{2V}{r}\right)$.

So, $\frac{dS}{dr} = \left(2\pi r - \frac{2V}{r^2}\right)$ and $\frac{d^2S}{dr^2} = \left(2\pi + \frac{4V}{r^3}\right)$.

For a maxima or minima, we have $\frac{dS}{dr} = 0$.

Now, $\frac{dS}{dr} = 0 \Rightarrow 2\pi r - \frac{2V}{r^2} = 0 \Rightarrow 2\pi r^3 = 2V = 2\pi r^2 h \Rightarrow r = h$.

$$\left(\frac{d^2S}{dr^2}\right)_{r=h} = \left(2\pi + \frac{4V}{h^3}\right) > 0.$$

Thus, $r = h$ is a point of minimum.

This shows that the total surface area of the cylinder is minimum when $h = r$, i.e., when the height of the cylinder is equal to the radius of the base.

EXAMPLE 16 *Show that the height of a cylinder which is open at the top, having a given surface and the greatest volume, is equal to the radius of its base.*

[CBSE 2004, 2010]

SOLUTION Let r be the radius, h the height and S the given surface area of an open cylinder. Then, $S = (\pi r^2 + 2\pi rh)$ or $h = \left(\frac{S - \pi r^2}{2\pi r}\right)$ … (i)

Let V be the volume of the cylinder.

Then, $V = \pi r^2 h = \pi r^2 \cdot \left(\frac{S - \pi r^2}{2\pi r}\right) = \left(\frac{rS - \pi r^3}{2}\right)$ [using (i)].

$\therefore \quad \frac{dV}{dr} = \left(\frac{S - 3\pi r^2}{2}\right)$ and $\frac{d^2V}{dr^2} = -3\pi r.$

Now, $\frac{dV}{dr} = 0 \Rightarrow S - 3\pi r^2 = 0 \Rightarrow \pi r^2 + 2\pi rh - 3\pi r^2 = 0 \Rightarrow h = r.$

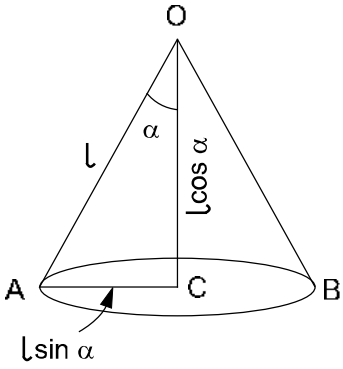
And, $\left(\frac{d^2V}{dr^2}\right)_{r=h} = -3\pi h < 0.$ Thus, $r = h$ is a point of maximum.

So, V is maximum when $r = h$, i.e., when the height of the cylinder is equal to the radius of its base.

EXAMPLE 17 *Show that the semivertical angle of a cone of maximum volume and of given slant height is* $\cos^{-1}\frac{1}{\sqrt{3}}$.

[CBSE 2008C, '14]

SOLUTION Let α be the semivertical angle of a cone of given slant height l. Then, height of the cone $= l\cos\alpha$, radius of the base $= l\sin\alpha$.



Let V be the volume of the cone. Then,

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(l^2\sin^2\alpha)(l\cos\alpha)$$

or $V = \frac{1}{3}\pi l^3 \sin^2\alpha\cos\alpha.$

$$\therefore \frac{dV}{d\alpha} = \frac{1}{3}\pi l^3(-\sin^3\alpha + 2\sin\alpha\cos^2\alpha) = \frac{1}{3}\pi l^3 \sin\alpha(2\cos^2\alpha - \sin^2\alpha).$$

And, $\frac{d^2V}{d\alpha^2} = \frac{1}{3}\pi l^3\cos\alpha(2\cos^2\alpha - \sin^2\alpha)$

$$+\frac{1}{3}\pi l^3\sin\alpha(-4\sin\alpha\cos\alpha - 2\sin\alpha\cos\alpha)$$

$$= \frac{1}{3}\pi l^3\cos^3\alpha(2 - 7\tan^2\alpha).$$

Now, for a maxima or minima, we have $\frac{dV}{d\alpha} = 0.$

Now, $\frac{dV}{d\alpha} = 0 \Rightarrow \frac{1}{3}\pi l^3\sin\alpha(2\cos^2\alpha - \sin^2\alpha) = 0$

$$\Rightarrow \sin\alpha = 0 \text{ or } 2\cos^2\alpha - \sin^2\alpha = 0$$

$$\Rightarrow \ \alpha = 0 \ \text{ or } \ \alpha = \tan^{-1}\sqrt{2}.$$

Neglecting $\alpha = 0$, we have $\alpha = \tan^{-1}\sqrt{2}$, i.e., $\tan\alpha = \sqrt{2}$.

Now, $\tan\alpha = \sqrt{2} \Rightarrow \sec^2\alpha = 3 \Rightarrow \cos\alpha = \dfrac{1}{\sqrt{3}} \Rightarrow \alpha = \cos^{-1}\dfrac{1}{\sqrt{3}}$

$$\therefore \quad \left[\frac{d^2V}{d\alpha^2}\right]_{\alpha=\tan^{-1}\sqrt{2}} = \frac{1}{3}\pi l^3 \cdot \left(\frac{1}{\sqrt{3}}\right)^3 (2-14) = -\frac{4\pi l^3}{3\sqrt{3}} < 0.$$

Thus, V is maximum when $\alpha = \cos^{-1}\dfrac{1}{\sqrt{3}}$, i.e., when the semivertical angle of the cone is $\cos^{-1}\dfrac{1}{\sqrt{3}}$.

EXAMPLE 18 *Show that the semivertical angle of a right circular cone of given surface area and maximum volume is* $\sin^{-1}(1/3)$.

SOLUTION Let r be the radius, l the slant height and h the height of the cone. Let S denote the surface area and V the volume of the cone.

Then, $S = (\pi r^2 + \pi r l) = \text{constant}.$

$$\therefore \quad l = \left(\frac{S}{\pi r} - r\right) \qquad \ldots \text{(i)}$$

Now, $V = \dfrac{1}{3}\pi r^2 h = \dfrac{1}{3}\pi r^2\sqrt{l^2 - r^2}.$

$$\therefore \quad V^2 = \frac{1}{9}\pi^2 r^4 (l^2 - r^2) = \frac{1}{9}\pi^2 r^4 \cdot \left\{\left(\frac{S}{\pi r} - r\right)^2 - r^2\right\} \quad \text{[using (i)]}$$

$$= \frac{1}{9}S(Sr^2 - 2\pi r^4).$$

Thus, $V^2 = \left(\dfrac{1}{9}S^2r^2 - \dfrac{2\pi}{9}Sr^4\right)$

$$\therefore \ 2V \cdot \frac{dV}{dr} = \left(\frac{2}{9}S^2 r - \frac{8\pi S}{9}r^3\right) = \frac{2rS}{9}(S - 4\pi r^2) \qquad \ldots \text{(ii)}$$

$$\therefore \frac{dV}{dr} = 0 \Rightarrow r = 0 \ \text{ or } \ (S - 4\pi r^2) = 0 \Rightarrow r^2 = \frac{S}{4\pi} \ (\text{neglecting } r = 0).$$

On differentiating (ii), we get $2\left(\dfrac{dV}{dr}\right)^2 + 2V \cdot \dfrac{d^2V}{dr^2} = \dfrac{1}{9}S(2S - 24\pi r^2).$

Putting $\dfrac{dV}{dr} = 0$ and $r^2 = \dfrac{S}{4\pi}$, we get

$$2V \cdot \frac{d^2V}{dr^2} = \frac{1}{9}S(2S - 6S) = -\frac{4}{9}S^2 < 0.$$

$\therefore$ when the volume is maximum, we have

$$r^2 = \frac{S}{4\pi} = \frac{(\pi r^2 + \pi r l)}{4\pi} \Rightarrow l = 3r.$$

Now, if α is the semivertical angle of the cone then

$$\frac{r}{l} = \sin\alpha \Rightarrow \frac{r}{3r} = \sin\alpha \Rightarrow \sin\alpha = \frac{1}{3} \Rightarrow \alpha = \sin^{-1}\left(\frac{1}{3}\right).$$

Hence, the semivertical angle of a right cone of a given surface and maximum volume is $\sin^{-1}(1/3)$.

EXAMPLE 19 *Show that the surface area of a closed cuboid with square base and given volume is minimum when it is a cube.* **[CBSE 2005]**

SOLUTION Let V be the fixed volume of a closed cuboid with length a, breadth a and height h.

Let S be its surface area.

Then, $V = (a \times a \times h)$ or $h = \dfrac{V}{a^2}$ … (i)

Now, $S = 2(a^2 + ah + ah) = 2(a^2 + 2ah) = 2\left(a^2 + \dfrac{2V}{a}\right)$ [using (i)]

i.e., $S = 2\left(a^2 + \dfrac{2V}{a}\right)$. $\therefore$ $\dfrac{dS}{da} = 2\left(2a - \dfrac{2V}{a^2}\right)$ and $\dfrac{d^2S}{da^2} = \left(4 + \dfrac{8V}{a^3}\right)$.

Now, $\dfrac{dS}{da} = 0 \Rightarrow V = a^3 \Rightarrow a \times a \times h = a^3 \Rightarrow h = a.$

Now, when $h = a$, we have $V = a^3$.

$\therefore$ $\left[\dfrac{d^2S}{da^2}\right]_{h=a} = \left(4 + \dfrac{8a^3}{a^3}\right) = 12 > 0.$

So, S is minimum when length = a, breadth = a and height = a, i.e., when it is a cube.

EXAMPLE 20 *Show that the height of a closed cylinder of given surface and maximum volume is equal to the diameter of its base.*

SOLUTION Let S be the fixed surface area of a closed cylinder of radius r and height h. Let V be its volume.

Then, $S = (2\pi r^2 + 2\pi rh) \Rightarrow h = \left(\dfrac{S - 2\pi r^2}{2\pi r}\right)$ … (i)

And, $V = \pi r^2 h = \pi r^2\left(\dfrac{S - 2\pi r^2}{2\pi r}\right) = \left(\dfrac{rS - 2\pi r^3}{2}\right)$ [using (i)].

$\Rightarrow$ $\dfrac{dV}{dr} = \left(\dfrac{S - 6\pi r^2}{2}\right)$ and $\dfrac{d^2V}{dr^2} = -6\pi r.$

Now, $\dfrac{dV}{dr} = 0 \Rightarrow (S - 6\pi r^2) = 0 \Rightarrow S = 6\pi r^2$

$\Rightarrow (2\pi r^2 + 2\pi rh) = 6\pi r^2 \Rightarrow h = 2r$, i.e., $r = \dfrac{1}{2}h.$

And, $\left[\dfrac{d^2V}{dr^2}\right]_{r=(h/2)} = -6\pi \times \dfrac{1}{2}h = -3\pi h < 0.$

So, V is maximum when $h = 2r$.

EXAMPLE 21 *A closed right circular cylinder has a volume of* 2156 *cubic units. What should be the radius of its base so that its total surface area may be maximum?*

SOLUTION Let r be the radius and h the height of the cylinder.

Then, $\pi r^2 h = 2156 \Rightarrow h = \left(\frac{2156}{\pi r^2}\right)$. Let S be its total surface area.

Then, $S = (2\pi r^2 + 2\pi rh) = \left(2\pi r^2 + 2\pi r \times \frac{2156}{\pi r^2}\right) = \left(2\pi r^2 + \frac{4312}{r}\right)$.

$\therefore \quad \frac{dS}{dr} = \left(4\pi r - \frac{4312}{r^2}\right)$ and $\frac{d^2S}{dr^2} = \left(4\pi + \frac{8624}{r^3}\right)$.

Now, $\frac{dS}{dr} = 0 \Rightarrow \left(4\pi r - \frac{4312}{r^2}\right) = 0$

$$\Rightarrow 4\pi r^3 = 4312 \Rightarrow r = \left(\frac{4312}{4\pi}\right)^{1/3} = \left(\frac{4312 \times 7}{4 \times 22}\right)^{1/3} = 7.$$

And, $\left[\frac{d^2S}{dr^2}\right]_{r=7} = \left(4\pi + \frac{8624}{343}\right) > 0$. $\therefore$ S is maximum when $r = 7$.

EXAMPLE 22 *Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is* $\frac{2R}{\sqrt{3}}$. *Find the volume of the largest cylinder inscribed in a sphere of radius R.* **[CBSE 2006, '09C, '12C, '13C]**

SOLUTION Let r be the radius and h the height of the inscribed cylinder $ABCD$. Let V be its volume.

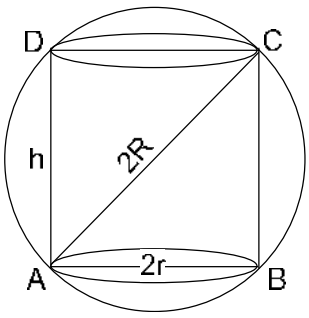
Then, $V = \pi r^2 h$ … (i)

Clearly, $AC = 2R$.

Also, $AC^2 = AB^2 + BC^2$

$\Rightarrow (2R)^2 = (2r)^2 + h^2$

$\Rightarrow r^2 = \frac{1}{4}(4R^2 - h^2)$ … (ii)



Using (ii) in (i), we get

$$V = \frac{\pi h}{4}(4R^2 - h^2) \Rightarrow \frac{dV}{dh} = \left(\pi R^2 - \frac{3}{4}\pi h^2\right) \text{ and } \frac{d^2V}{dh^2} = -\frac{3}{2}\pi h.$$

For a maxima or minima, we have $(dV/dh) = 0$.

Now, $\frac{dV}{dh} = 0 \Rightarrow \pi R^2 - \frac{3}{4}\pi h^2 = 0 \Rightarrow h = \frac{2R}{\sqrt{3}}$.

$$\therefore \quad \left[\frac{d^2V}{dh^2}\right]_{h=(2R/\sqrt{3})} = -\frac{3}{2}\pi \times \frac{2R}{\sqrt{3}} = -\pi R\sqrt{3} < 0.$$

So, V is maximum when $h = \frac{2R}{\sqrt{3}}$.

Hence, the height of the cylinder of maximum volume is $\frac{2R}{\sqrt{3}}$.

Largest volume of the cylinder $= \pi \times \frac{1}{4}\left[4R^2 - \frac{4R^2}{3}\right] \times \frac{2R}{\sqrt{3}} = \frac{4\pi R^3}{3\sqrt{3}}$.

EXAMPLE 23 *Show that the cone of greatest volume which can be inscribed in a given sphere is such that three times its altitude is twice the diameter of the sphere. Find the volume of the largest cone inscribed in a sphere of radius R.* **[CBSE 2010C]**

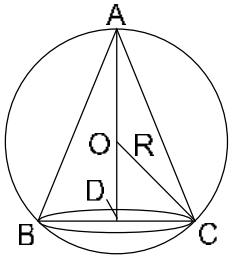
SOLUTION Let R be the radius of the given sphere with centre O, and let V be the volume of the inscribed cone, h be its height and r be the radius of the base.

In the given figure, we have $OD = AD - AO = (h - R)$.

$\therefore \quad R^2 = (h-R)^2 + r^2$ or $r^2 = h(2R - h)$ … (i)

Now, $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi h^2(2R - h)$ [using (i)].

$\therefore \quad \frac{dV}{dh} = \frac{1}{3}\pi h(4R - 3h)$, and $\frac{d^2V}{dh^2} = \left(\frac{4}{3}\pi R - 2\pi h\right)$.



For a maxima or minima, we have $\frac{dV}{dh} = 0$.

Now, $\frac{dV}{dh} = 0 \Rightarrow \frac{1}{3}\pi h(4R - 3h) = 0$

$\Rightarrow h = 0$ or $(4R - 3h) = 0 \Rightarrow h = \frac{4}{3}R \quad [\because h \neq 0]$.

And, $\left[\frac{d^2V}{dh^2}\right]_{h=(4/3)R} = -\frac{4\pi R}{3} < 0.$

So, V is maximum when $h = \frac{4}{3}R$, i.e., when $3h = 2(2R)$

i.e., 3 times the height = 2 times the diameter.

Volume of the largest cone $= \frac{1}{3}\pi \times \frac{16R^2}{9} \times \left(2R - \frac{4R}{3}\right) = \frac{32\pi R^3}{81}$.

EXAMPLE 24 *Prove that the perimeter of a right-angled triangle of given hypotenuse is maximum when the triangle is isosceles.*

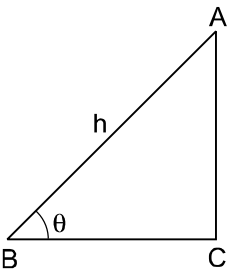
SOLUTION Let h be the hypotenuse and θ be the angle between the hypotenuse and the base.

Then, base $= h\cos\theta$ and altitude $= h\sin\theta$.

Let P be the perimeter.

Then, $P = h + h\cos\theta + h\sin\theta$.

$\therefore \quad \frac{dP}{d\theta} = -h\sin\theta + h\cos\theta = h(\cos\theta - \sin\theta)$.



And, $\dfrac{d^2P}{d\theta^2} = h(-\sin\theta - \cos\theta) = -h(\sin\theta + \cos\theta).$

Now, $\dfrac{dP}{d\theta} = 0 \Rightarrow h(\cos\theta - \sin\theta) = 0$

$\Rightarrow \cos\theta = \sin\theta$, i.e., $\tan\theta = 1 \Rightarrow \theta = (\pi/4).$

Also, $\left[\dfrac{d^2P}{d\theta^2}\right]_{\theta=(\pi/4)} = -h\left[\sin\dfrac{\pi}{4} + \cos\dfrac{\pi}{4}\right] = -h\sqrt{2} < 0.$

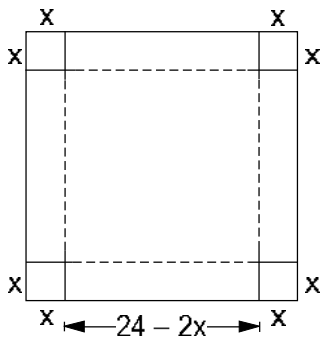
$\therefore$ P is maximum when $\theta = (\pi/4)$.

In this case, base $= h\cos\dfrac{\pi}{4} = \dfrac{h}{\sqrt{2}}$ and altitude $= h\sin\dfrac{\pi}{4} = \dfrac{h}{\sqrt{2}}$.

$\therefore$ base = altitude, and hence the triangle is isosceles.

EXAMPLE 25 *An open box is to be made out of a piece of cardboard measuring* (24 cm $\times$ 24 cm) *by cutting off equal squares from the corners and turning up the sides. Find the height of the box when it has maximum volume.* **[CBSE 2004]**

SOLUTION Let the length of the side of each square cut off from the corners be x cm.



Then, height of the box $= x$ cm.

$\therefore$ $V = (24 - 2x)^2 \times x = 4x^3 - 96x^2 + 576x$

$\Rightarrow$ $\dfrac{dV}{dx} = 12(x^2 - 16x + 48)$

and, $\dfrac{d^2V}{dx^2} = 24(x - 8).$

Now, $\dfrac{dV}{dx} = 0 \Rightarrow x^2 - 16x + 48 = 0$, i.e., $(x - 12)(x - 4) = 0$

$\Rightarrow x = 4 \quad [\because x \neq 12]$

$\left[\dfrac{d^2V}{dx^2}\right]_{x=4} = -96 < 0.$ $\therefore$ V is maximum at $x = 4$.

Hence, the volume of the box is maximum when its height is 4 cm.

EXERCISE 11F

1. Find two positive numbers whose product is 49 and the sum is minimum.
2. Find two positive numbers whose sum is 16 and the sum of whose squares is minimum.
3. Divide 15 into two parts such that the square of one number multiplied with the cube of the other number is maximum.

4. Divide 8 into two positive parts such that the sum of the square of one and the cube of the other is minimum.
5. Divide a into two parts such that the product of the pth power of one part and the qth power of the second part may be maximum.
6. The rate of working of an engine is given by

$$R = 15v + \frac{6000}{v}, \text{ where } 0 < v < 30$$

and v is the speed of the engine. Show that R is the least when $v = 20$.

7. Find the dimensions of the rectangle of area 96 cm^2 whose perimeter is the least. Also, find the perimeter of the rectangle.
8. Prove that the largest rectangle with a given perimeter is a square.
9. Given the perimeter of a rectangle, show that its diagonal is minimum when it is a square.
10. Show that a rectangle of maximum perimeter which can be inscribed in a circle of radius a is a square of side $\sqrt{2}\,a$.
11. The sum of the perimeters of a square and a circle is given. Show that the sum of their areas is least when the side of the square is equal to the diameter of the circle.
12. Show that the right triangle of maximum area that can be inscribed in a circle is an isosceles triangle.
13. Prove that the perimeter of a right-angled triangle of given hypotenuse is maximum when the triangle is isosceles.
14. The perimeter of a triangle is 8 cm. If one of the sides of the triangle be 3 cm, what will be the other two sides for maximum area of the triangle?
15. A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 metres. Find the dimensions of the window to admit maximum light through it.

[CBSE 2000, '02, '06]

16. A square piece of tin of side 12 cm is to be made into a box without a lid by cutting a square from each corner and folding up the flaps to form the sides. What should be the side of the square to be cut off so that the volume of the box is maximum? Also, find this maximum volume.
17. An open box with a square base is to be made out of a given cardboard of area c^2 (square) units. Show that the maximum volume of the box is $\frac{c^3}{6\sqrt{3}}$ (cubic) units. **[CBSE 2001]**
18. A cylindrical can is to be made to hold 1 litre of oil. Find the dimensions which will minimize the cost of the metal to make the can.
19. Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ times the radius of the base. **[CBSE 2007]**

20. Find the radius of a closed right circular cylinder of volume $100\ \text{cm}^3$ which has the minimum total surface area.

21. Show that the height of a closed cylinder of given volume and the least surface area is equal to its diameter.

22. Prove that the volume of the largest cone that can be inscribed in a sphere is $\frac{8}{27}$ of the volume of the sphere. **[CBSE 2004, '04C, '05C]**

23. Which fraction exceeds its pth power by the greatest number possible?

24. Find the point on the curve $y^2 = 4x$ which is nearest to the point $(2, -8)$.

25. A right circular cylinder is inscribed in a cone. Show that the curved surface area of the cylinder is maximum when the diameter of the cylinder is equal to the radius of the base of the cone. **[CBSE 2006C]**

26. Show that the surface area of a closed cuboid with square base and given volume is minimum when it is a cube.

27. A rectangle is inscribed in a semicircle of radius r with one of its sides on the diameter of the semicircle. Find the dimensions of the rectangle so that its area is maximum. Find also this area. **[CBSE 2004]**

28. Two sides of a triangle have lengths a and b and the angle between them is θ. What value of θ will maximize the area of the triangle? **[CBSE 2002C]**

29. Show that the maximum volume of the cylinder which can be inscribed in a sphere of radius $5\sqrt{3}$ cm is $(500\pi)\ \text{cm}^3$. **[CBSE 2004, 05C]**

30. A square tank of capacity 250 cubic metres has to be dug out. The cost of the land is ₹ 50 per square metre. The cost of digging increases with the depth and for the whole tank, it is ₹ $(400 \times h^2)$, where h metres is the depth of the tank. What should be the dimensions of the tank so that the cost is minimum? **[CBSE 2003]**

31. A square piece of tin of side 18 cm is to be made into a box without the top, by cutting a square piece from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum? Also, find the maximum volume of the box. **[CBSE 2003]**

32. An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of the material will be least when the depth of the tank is half of its width. **[CBSE 2003, '07C]**

33. A wire of length 36 cm is cut into two pieces. One of the pieces is turned in the form of a square and the other in the form of an equilateral triangle. Find the length of each piece so that the sum of the areas of the two be minimum. **[CBSE 2005]**

34. Find the largest possible area of a right-angled triangle whose hypotenuse is 5 cm. **[CBSE 2006C]**

ANSWERS (EXERCISE 11F)

1. 7, 7 **2.** 8, 8 **3.** 6, 9 **4.** 6, 2 **5.** $\frac{ap}{p+q}, \frac{aq}{p+q}$

7. length $= 4\sqrt{6}$ cm, breadth $= 4\sqrt{6}$ cm, perimeter $= 16\sqrt{6}$ cm

14. $b = c = 2.5$ cm **15.** breadth $= \frac{20}{(\pi+4)}$ m, height $= \frac{10}{(\pi+4)}$ m

16. 2 cm, 128 cm^3 **18.** $r = \left(\frac{500}{\pi}\right)^{1/3}$ cm, $h = \frac{1000}{\pi^{1/3}(500)^{2/3}}$ cm **20.** $\left(\frac{50}{\pi}\right)^{1/3}$ cm

23. $\left(\frac{1}{p}\right)^{1/(p-1)}$ **24.** $(4, -4)$

27. length $= r\sqrt{2}$, breadth $= \frac{r\sqrt{2}}{2}$, area $= r^2$ sq units **28.** $\frac{\pi}{2}$

30. Side = 10 cm, depth = 2.5 m **31.** 3 cm, 432 cm^3

33. $\frac{144\sqrt{3}}{4\sqrt{3}+9}$ cm, $\frac{324}{4\sqrt{3}+9}$ cm **34.** $\frac{25}{4}$ sq cm

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 11F)

7. Let P be the fixed perimeter and x, y be the sides.

Then, $96 = xy \Rightarrow y = \frac{96}{x}$.

$P = 2(x + y) \Rightarrow P = 2\left(x + \frac{96}{x}\right)$.

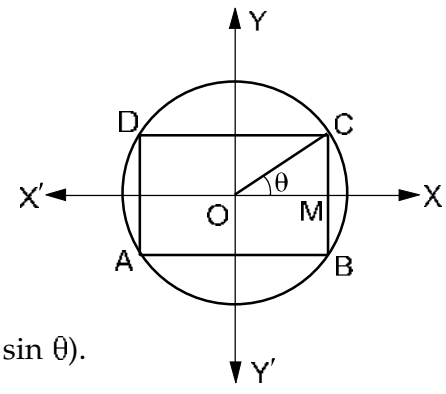
$P = 2(x + y) \Rightarrow y = \frac{1}{2}(P - 2x)$. $\therefore A = xy \Rightarrow A = \frac{1}{2}x(P - 2x)$.

9. $P = 2(x + y) \Rightarrow y = \frac{1}{2}(P - 2x)$.

$\therefore \quad z = D^2 = x^2 + y^2 \Rightarrow z = x^2 + \frac{1}{4}(P - 2x)^2$.

Now, D is minimum when z is minimum.

10. Let $ABCD$ be the rectangle inscribed in a circle of radius a and with centre O. Join OC. Let $\angle COX = \theta$. Then, the coordinates of C are $(a\cos\theta, a\sin\theta)$.



$\therefore \quad OM = a\cos\theta$ and $MC = a\sin\theta$.

$BC = 2MC = 2a$ and $CD = 2OM = 2a\cos\theta$.

$\therefore \quad P = 2a(\cos\theta + \sin\theta)$.

So, $\frac{dP}{d\theta} = 2a(-\sin\theta + \cos\theta)$ and $\frac{d^2P}{d\theta^2} = -2a(\cos\theta + \sin\theta)$.

$\frac{dP}{d\theta} = 0 \Rightarrow \theta = \frac{\pi}{4}$ and at this value of θ, $\frac{d^2P}{d\theta} < 0$.

So, P is maximum at $\theta = (\pi/4)$.

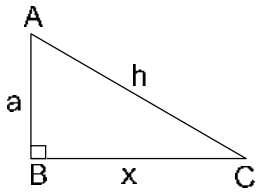
Then, $BC = \sqrt{2}\, a = CD$.

11. Let $S = 4x + 2\pi y$. Then, $y = \frac{(S - 4x)}{2\pi}$.

Let $A = x^2 + \pi y^2$. Then, $A = x^2 + \frac{(S - 4x)^2}{4\pi}$.

13. Let ABC be the right triangle with given hypotenuse h.

Let, base $BC = x$ and altitude $BA = a$.



Then, $h^2 = x^2 + a^2 \Rightarrow a = \sqrt{h^2 - x^2}$.

$\therefore \quad P = a + x + h \Rightarrow P = \sqrt{(h^2 - x^2)} + x + h$.

So, $\frac{dP}{dx} = \frac{-x}{\sqrt{h^2 - x^2}} + 1$ and $\frac{d^2P}{dx^2} = \frac{-h^2}{(h^2 - x^2)^{3/2}}$.

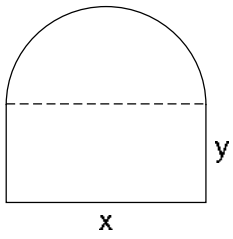
Now, $\frac{dP}{dx} = 0 \Rightarrow x = \frac{h}{\sqrt{2}}$.

And for $x = \frac{h}{\sqrt{2}}$, $\frac{d^2P}{dx^2} = \frac{-2^{3/2}}{h} < 0$.

$\therefore$ P is maximum when $x = \frac{h}{\sqrt{2}}$ and $a = \frac{h}{\sqrt{2}}$, i.e., when $x = a$.

15. Let the length and breadth of the rectangle be x and y metres respectively. Then, radius of the semicircle $= (x/2)$ metres.

So, the perimeter of the window is given by



$10 = x + 2y + \frac{\pi x}{2} \Rightarrow y = 5 - \left(\frac{2 + \pi}{4}\right)x$.

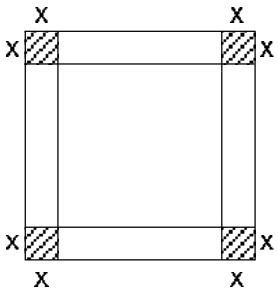
$\therefore \quad A = xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2 \Rightarrow A = x\left[5 - \frac{(2 + \pi)}{4} \cdot x\right] + \frac{\pi x^2}{8}$

$\therefore \quad \frac{dA}{dx} = \left(5 - x - \frac{\pi x}{4}\right)$ and $\frac{d^2A}{dx^2} = -\left(1 + \frac{\pi}{4}\right) < 0$.

Now, $\frac{dA}{dx} = 0 \Rightarrow x = \frac{20}{(\pi + 4)}$.

$\therefore$ A is maximum when $x = \frac{20}{(\pi + 4)}$ and $y = \frac{10}{(\pi + 4)}$.

16. Let the length cut off from each corner be x cm. Then, length of the box $= (12 - 2x)$ cm = breadth and height $= x$ cm.



$\therefore \quad V = (12 - 2x)^2 \times x \Rightarrow V = 4x^3 - 48x^2 + 144x$.

$\therefore \quad \frac{dV}{dx} = 12(x - 6)(x - 2)$ and $\frac{d^2V}{dx^2} = 24(x - 4)$.

Then, $\frac{dV}{dx} = 0 \Rightarrow x = 2$ [$\because$ $x = 6$ is not possible].

Also at $x = 2$, $\frac{d^2V}{dx^2} < 0$.

$\therefore$ V is maximum when $x = 2$ and maximum $V = [(12 - 4)^2 \times 2]$ cm^3.

17. Let each side of the base be a and height be h. Then,

$$c^2 = (a^2 + 4ah) \Rightarrow h = \frac{(c^2 - a^2)}{4a}.$$

So, $V = a^2 \times h \Rightarrow V = \frac{(c^2 a - a^3)}{4}.$

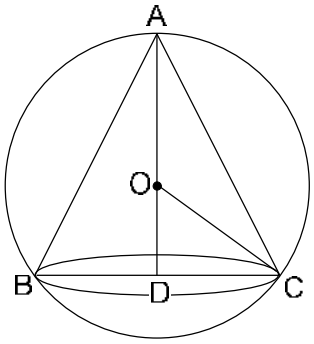
$$\therefore \quad \frac{dV}{da} = \frac{(c^2 - 3a^2)}{4} \text{ and } \frac{d^2V}{da^2} = -\frac{3}{2}a < 0.$$

Thus, V is maximum when $a^2 = \frac{c^2}{3}$ and $h = \frac{c}{2\sqrt{3}}$.

18. $\pi r^2 h = 1000 \Rightarrow h = \frac{1000}{\pi r^2}$ [$\because$ $V = 1$ litre $= 1000$ cm^3].

$$\therefore \quad S = (2\pi r^2 + 2\pi rh) \Rightarrow S = 2\left(\pi r^2 + \frac{1000}{r}\right).$$

22. Let the radius of the given sphere be a. Let V be the volume of the inscribed cone, h be its height and r be its radius.



Then, $V = \frac{1}{3}\pi r^2 h.$

Now, $OD = (AD - OA) = (h - a).$

$$OC^2 = OD^2 + DC^2 \Rightarrow a^2 = (h - a)^2 + r^2$$
$$\Rightarrow r^2 = h(2a - h).$$

$$\therefore \quad V = \frac{1}{3}\pi h^2(2a - h) = \left[\frac{2\pi ah^2}{3} - \frac{1}{3}\pi h^3\right].$$

$$\therefore \quad \frac{dV}{dh} = \left(\frac{4\pi ah}{3} - \pi h^2\right) \text{ and } \frac{d^2V}{dh^2} = \left(\frac{4\pi a}{3} - 2\pi h\right).$$

So, $\frac{dV}{dh} = 0 \Rightarrow h = \frac{4a}{3}$ and $\left[\frac{d^2V}{dh^2} \text{ at } h = \frac{4a}{3}\right] = -\frac{4\pi a}{3} < 0.$

$\therefore$ V is maximum when $h = \frac{4a}{3}$ and $r^2 = \frac{8a^2}{9}$.

Maximum volume $= \frac{1}{3}\pi \times \frac{8a^2}{9} \times \frac{4a}{3} = \frac{8}{27} \times \left(\frac{4}{3}\pi a^3\right).$

23. Let the required fraction be x. Let $y = (x - x^p)$. Then,

$$\frac{dy}{dx} = 0 \Rightarrow x = \left(\frac{1}{p}\right)^{1/(p-1)}. \qquad \frac{d^2y}{dx^2} \text{ at this point } = -p(p-1)\left(\frac{1}{p}\right)^{\left(\frac{p-2}{p-1}\right)} < 0.$$

$\therefore$ y is maximum when $x = \left(\frac{1}{p}\right)^{1/(p-1)}$.

24. Let the required point be (x, y). Then,

$$D = (x - 2)^2 + (y + 8)^2 \Rightarrow D = x^2 + y^2 - 4x + 16y + 68$$

$$\Rightarrow D = \frac{y^4}{16} + 16y + 68 \quad \left[\because x = \frac{y^2}{4}\right]$$

$$\Rightarrow \frac{dD}{dy} = \left(\frac{4y^3}{16} + 16\right) \text{ and } \frac{d^2D}{dy^2} = \frac{3}{4}y^2.$$

Now, $\frac{dD}{dy} = 0 \Rightarrow y^3 = -64 \Rightarrow y = -4.$

$\left[\frac{d^2D}{dy^2}\right]_{y=-4} = \frac{3}{4} \times 16 = 12 > 0.$

$\therefore$ D is minimum when $y = -4$.

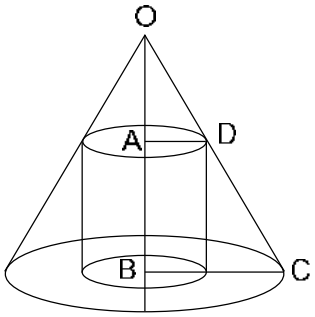
At this value, we have $x = 4$.

Hence, the required point is $(4, -4)$.

25. Let r_1 be the radius of the cone and h_1 be its height and let r be the radius and h the height of the inscribed cylinder.

Clearly, $\triangle OAD$ and $\triangle OBC$ are similar.

$\therefore$ $\frac{r}{r_1} = \frac{h_1 - h}{h_1} \Rightarrow r = \frac{r_1}{h_1}(h_1 - h)$ … (i)



Let S be the curved surface area of the cylinder.

Then, $S = 2\pi rh \Rightarrow S = \frac{2\pi r_1}{h_1}(h_1 - h)h$

$\Rightarrow \frac{dS}{dh} = \frac{2\pi r_1}{h_1}(h_1 - 2h)$ and $\frac{d^2S}{dh^2} = \frac{-4\pi r_1}{h_1} < 0.$

Now, $\frac{dS}{dh} = 0 \Leftrightarrow h_1 = 2h$. Also, $\frac{d^2S}{dh^2} < 0.$

So, S is maximum when $h_1 = 2h$.

$\therefore$ $\frac{r}{r_1} = \frac{2h - h}{2h} = \frac{1}{2} \Leftrightarrow r_1 = 2r =$ diameter of the cylinder.

26. Let V be the volume, x the side of the square base and h the height of the cuboid.

Then, $V = x^2h \Rightarrow h = \frac{V}{x^2}$ … (i)

$\therefore$ $S = 2(x^2 + xh + xh) \Rightarrow S = 2x^2 + 4xh.$

$\therefore$ $S = 2x^2 + \frac{4V}{x}$ [using (i)]

$\Rightarrow$ $\frac{dS}{dx} = 4x - \frac{4V}{x^2}$ and $\frac{d^2S}{dx^2} = 4 + \frac{8V}{x^3}.$

Now, $\frac{dS}{dx} = 0 \Rightarrow x^3 = V.$

For this value of x, $\frac{d^2S}{dx^2} = 4 + \frac{8V}{V} = 4 + 8 = 12 > 0.$

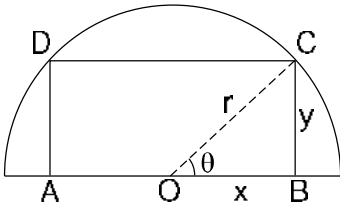
$\therefore$ surface area is minimum when $x^3 = V$.

Also, $h = \frac{V}{x^2} = \frac{x^3}{x^2} = x.$

$\therefore$ S is minimum when the cuboid is a cube.

27. Let $ABCD$ be the rectangle of length $2x$ and breadth y, inscribed in a semicircle of radius r and centre O. Let $\angle BOC = \theta$.

Then, $\frac{y}{r} = \sin\theta$ and $\frac{x}{r} = \cos\theta.$



$\therefore$ area of the rectangle is given by

$A = 2xy = r^2 \sin 2\theta$

$\Rightarrow$ $\frac{dA}{d\theta} = 2r^2 \cos 2\theta$ and $\frac{d^2A}{d\theta^2} = -4r^2 \sin 2\theta.$

Now, $\frac{dA}{d\theta} = 0 \Rightarrow \cos 2\theta = 0 \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$.

For this value of θ, we have $\frac{d^2A}{d\theta^2} = -4r^2 < 0$.

$\therefore$ area is maximum when $2x = 2r\cos\frac{\pi}{4} = r\sqrt{2}$ and $y = r\sin\frac{\pi}{4} = \frac{r\sqrt{2}}{2}$.

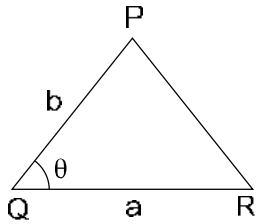
Maximum area $= 2xy = \frac{r\sqrt{2} \times r\sqrt{2}}{2} = r^2$ sq units.

28. Let A be the area of $\triangle PQR$.

Then, $A = \frac{1}{2}ab\sin\theta$

$\Rightarrow \frac{dA}{d\theta} = \frac{1}{2}ab\cos\theta$

$\Rightarrow \frac{d^2A}{d\theta^2} = -\frac{1}{2}ab\sin\theta < 0$.



For maxima or minima, we have

$\frac{dA}{d\theta} = 0 \Leftrightarrow \frac{1}{2}ab\cos\theta = 0 \Leftrightarrow \cos\theta = 0 \Leftrightarrow \theta = \frac{\pi}{2}$.

$\therefore$ A is maximum, when $\theta = \frac{\pi}{2}$.

30. Let the side of the square tank be x metres.

Then, total cost is given by $C = 50x^2 + 400h^2$ … (i)

Also, $x^2h = 250 \Rightarrow h = \frac{250}{x^2}$ … (ii)

$\therefore$ $C = 50x^2 + \frac{400 \times 62500}{x^4}$

$\Rightarrow \frac{dC}{dx} = (100x - 10^8 \cdot x^{-5})$ and $\frac{d^2C}{dx^2} = \left(100 + \frac{10^8 \times 5}{x^6}\right)$.

Now, $\frac{dC}{dx} = 0 \Leftrightarrow x = 10$ and $\left[\frac{d^2C}{dx^2}\right]_{(x=10)} = 600 > 0$.

$\therefore$ cost is minimum when $x = 10$ and $h = \left(\frac{250}{100}\right)$ m = 2.5 m.

Hence, for minimum cost, the tank must have a square base of side 10 m and depth 2.5 m.

31. Let the side of the square piece cut from each corner of the given square plate of side 18 cm be x cm.

Then, the dimensions of the open box are $(18 - 2x)$ cm, $(18 - 2x)$ cm and x cm.

$\therefore$ $V = (18 - 2x)^2 \times x$

$\Rightarrow$ $V = 4x^3 - 72x^2 + 324x$ … (i)

$\Rightarrow \frac{dV}{dx} = 12(x^2 - 12x + 27)$ and $\frac{d^2V}{dx^2} = 12(2x - 12)$.

Now, $\frac{dV}{dx} = 0 \Leftrightarrow x^2 - 12x + 27 = 0 \Leftrightarrow (x-3)(x-9) = 0 \Leftrightarrow x = 3 \quad [\because x \neq 9]$

and $\left[\frac{d^2V}{dx^2}\right]_{(x=3)} = 12(2 \times 3 - 12) = -72 < 0.$

$\therefore$ V is maximum at $x = 3$ cm, and
maximum volume $= [4 \times 3^3 - 72 \times 3^2 + 324 \times 3]\ \text{cm}^3 = 432\ \text{cm}^3$.

32. Let x be the side of the square base and h be the depth.
Then, $V = x^2h$.
Let the cost per square metre be ₹ p. Then,

$$C = (x^2 + 4xh)\,p \Rightarrow C = \left(x^2 + 4x \times \frac{V}{x^2}\right)p \Rightarrow C = \left(x^2 + \frac{4V}{x}\right)p.$$

$\therefore$ $\frac{dC}{dx} = \left(2x - \frac{4V}{x^2}\right)p$ and $\frac{d^2C}{dx^2} = \left(2 + \frac{8V}{x^3}\right)p.$

Now, $\frac{dC}{dx} = 0 \Leftrightarrow \left(2x - \frac{4V}{x^2}\right) = 0 \Leftrightarrow x = (2V)^{1/3}$

And, $\left[\frac{d^2C}{dx^2}\right]_{x=(2V)^{1/3}} = p\left(2 + \frac{8V}{2V}\right) = 6p > 0.$

$\therefore$ C is minimum, when $x = (2V)^{1/3}$.

Then, $h = \frac{V}{x^2} = \frac{1}{2}(2V)^{1/3} = \frac{1}{2}x.$

Hence, for minimum cost, the depth of the tank should be equal to half of the side of its square base.

33. Let the perimeter of the square be x cm.
Then the perimeter of the triangle is $(36 - x)$ cm.

$\therefore$ side of the square $= \frac{x}{4}$ cm.

And, side of the triangle $= \frac{1}{3}(36 - x)$ cm.

$$\therefore A = \frac{x^2}{16} + \frac{\sqrt{3}}{4}\left(12 - \frac{x}{3}\right)^2 = \frac{x^2}{16} + \frac{\sqrt{3}}{4}\left(144 + \frac{x^2}{9} - 8x\right)$$

$$\Rightarrow A = \left(\frac{\sqrt{3}}{36} + \frac{1}{16}\right)x^2 - 2\sqrt{3}x + 36\sqrt{3}$$

$$\Rightarrow \frac{dA}{dx} = \frac{(4\sqrt{3}+9)}{144} \times 2x - 2\sqrt{3} \text{ and } \frac{d^2A}{dx^2} = \frac{4\sqrt{3}+9}{72} > 0$$

$$\therefore \frac{dA}{dx} = 0 \Leftrightarrow x = \frac{144\sqrt{3}}{(4\sqrt{3}+9)} \text{ cm.}$$

34. Consider a right-angled $\triangle ABC$ in which hyp. $AC = 5$ cm and let $\angle BAC = \theta$.
Then, $AB = 5\cos\theta$ and $BC = 5\sin\theta$.

$\therefore$ area, $A = \frac{1}{2}AB \times BC = \frac{1}{2} \times 5\cos\theta \times 5\sin\theta$

$$\Rightarrow A = \frac{25}{4}\sin 2\theta$$

$$\Rightarrow \frac{dA}{d\theta} = \frac{25}{2}\cos 2\theta \text{ and } \frac{d^2A}{d\theta^2} = -25\sin 2\theta.$$

Now, $\frac{dA}{d\theta} = 0 \Rightarrow \cos 2\theta = 0 \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$.

$$\left.\frac{d^2A}{d\theta^2}\right]_{\theta=\frac{\pi}{4}} = -25\sin\left(\frac{\pi}{2}\right) = -25 < 0.$$

$\therefore \quad \theta = \frac{\pi}{4}$ is a point of maxima.

$$\text{Maximum area} = \frac{25}{4} \times \sin\left(2 \times \frac{\pi}{4}\right) = \frac{25}{4} \text{ sq cm.}$$

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5. Increasing and Decreasing Functions

INCREASING FUNCTION *A function f(x) defined on* $]a, b[$ *is said to be increasing if*

$$x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2) \text{ for all } x_1, x_2 \in]a, b[$$

or $x_1 > x_2 \Rightarrow f(x_1) \geq f(x_2)$ *for all* $x_1, x_2 \in]a, b[$.

STRICTLY INCREASING FUNCTION *A function f(x) defined on* $]a, b[$ *is said to be strictly increasing if*

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \text{ for all } x_1, x_2 \in]a, b[$$

or $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$ *for all* $x_1, x_2 \in]a, b[$.

DECREASING FUNCTION *A function f(x) defined on* $]a, b[$ *is said to be decreasing if*

$$x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2) \text{ for all } x_1, x_2 \in]a, b[$$

or $x_1 > x_2 \Rightarrow f(x_1) \leq f(x_2)$ *for all* $x_1, x_2 \in]a, b[$.

STRICTLY DECREASING FUNCTION *A function f(x) defined on* $]a, b[$ *is said to be strictly decreasing if*

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2) \text{ for all } x_1, x_2 \in]a, b[$$

or $x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$ *for all* $x_1, x_2 \in]a, b[$.

EXAMPLE 1 *Show that* $f(x) = 3x + 5$ *is a strictly increasing function on R.*

SOLUTION Let $x_1, x_2 \in R$ such that $x_1 < x_2$. Then,

$$\begin{aligned} x_1 < x_2 &\Rightarrow 3x_1 < 3x_2 \\ &\Rightarrow 3x_1 + 5 < 3x_2 + 5 \\ &\Rightarrow f(x_1) < f(x_2). \end{aligned}$$

Thus, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in R$.
Hence, $f(x)$ is strictly increasing on R.

EXAMPLE 2 *Show that the function $f(x) = e^x$ is strictly increasing on R.*

SOLUTION Let $x_1, x_2 \in R$ such that $x_1 > x_2$. Then,

$$x_1 > x_2 \Rightarrow e^{x_1} > e^{x_2} \quad [\because e > 1 \text{ and } x_1 > x_2 \Rightarrow e^{x_1} > e^{x_2}]$$
$$\Rightarrow f(x_1) > f(x_2).$$

Thus, $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in R$.
Hence, $f(x)$ is strictly increasing on R.

EXAMPLE 3 *Show that $f(x) = e^{-x}$ is a strictly decreasing function on R.*

SOLUTION Let $x_1, x_2 \in R$ such that $x_1 > x_2$. Then,

$$x_1 > x_2 \Rightarrow e^{x_1} > e^{x_2} \quad [\because e > 1 \text{ and } x_1 > x_2 \Rightarrow e^{x_1} > e^{x_2}]$$
$$\Rightarrow \frac{1}{e^{x_1}} < \frac{1}{e^{x_2}}$$
$$\Rightarrow e^{-x_1} < e^{-x_2}$$

Thus, $x_1 > x_2 \Rightarrow e^{-x_1} < e^{-x_2}$ for all $x_1, x_2 \in R$.
Hence, $f(x) = e^{-x}$ is strictly decreasing on R.

EXAMPLE 4 *If a is a real number greater than 1, show that the function $f(x) = a^x$ is strictly increasing on R.*

SOLUTION Let $x_1, x_2 \in R$ such that $x_1 > x_2$. Then,

$$x_1 > x_2 \Rightarrow a^{x_1} > a^{x_2} \quad [\because a > 1 \text{ and } x_1 > x_2 \Rightarrow a^{x_1} > a^{x_2}]$$
$$\Rightarrow f(x_1) > f(x_2).$$

Thus, $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in R$.
Hence, $f(x)$ is strictly increasing on R.

EXAMPLE 5 *If a is a real number such that $0 < a < 1$, show that the function $f(x) = a^x$ is strictly decreasing on R.*

SOLUTION Let $x_1, x_2 \in R$ such that $x_1 < x_2$. Then,

$$x_1 < x_2 \Rightarrow a^{x_1} > a^{x_2} \quad [\because 0 < a < 1 \text{ and } x_1 < x_2 \Rightarrow a^{x_1} > a^{x_2}]$$
$$\Rightarrow f(x_1) > f(x_2).$$

Thus, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in R$.
Hence, $f(x)$ is strictly decreasing on R.

THEOREM 1 *Let $f(x)$ be a function continuous on $[a, b]$ and differentiable on $]a, b[$. Then,*

$$f'(x) > 0 \text{ for all } x \in]a, b[\Rightarrow f(x) \text{ is increasing on }]a, b[.$$

PROOF Let $f'(x) > 0$ for all $x \in]a, b[$.
Let $x_1, x_2 \in]a, b[$ such that $x_1 < x_2$.
Then, clearly $f(x)$ is continuous on $[x_1, x_2]$ and differentiable on $]x_1, x_2[$.
So, by the mean-value theorem, there exists a real number $c \in]x_1, x_2[$ such that $f'(c) = \dfrac{f(x_2) - f(x_1)}{(x_2 - x_1)}$.

Since $f'(x) > 0$ for all $x \in]a, b[$, in particular, $f'(c) > 0$.

Now, $f'(c) > 0 \Rightarrow \dfrac{f(x_2) - f(x_1)}{(x_2 - x_1)} > 0$

$\Rightarrow f(x_2) - f(x_1) > 0 \quad [\because x_2 - x_1 > 0 \text{ when } x_1 < x_2]$

$\Rightarrow f(x_2) > f(x_1) \Rightarrow f(x_1) < f(x_2)$.

Thus, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.
Hence, $f(x)$ is an increasing function on $]a, b[$.

THEOREM 2 *Let $f(x)$ be a function, continuous on $[a, b]$ and differentiable on $]a, b[$. Then, $f'(x) < 0$ for all $x \in]a, b[\Rightarrow f(x)$ is decreasing on $]a, b[$.*

PROOF Let it be given that $f'(x) < 0$ for all $x \in]a, b[$.
Let $x_1, x_2 \in]a, b[$ such that $x_1 < x_2$.
Then, $f(x)$ is continuous on $[x_1, x_2]$ and differentiable on $]x_1, x_2[$.
So, by the mean-value theorem, there exists a real number $c \in]x_1, x_2[$ such that $f'(c) = \dfrac{f(x_2) - f(x_1)}{(x_2 - x_1)}$.

Since, $f'(x) < 0$ for all $x \in]a, b[$, in particular, $f'(c) < 0$.

Now, $f'(c) < 0 \Rightarrow \dfrac{f(x_2) - f(x_1)}{(x_2 - x_1)} < 0$

$\Rightarrow f(x_2) - f(x_1) < 0 \quad [\because \quad x_2 - x_1 > 0 \text{ when } x_1 < x_2]$

$\Rightarrow f(x_2) < f(x_1) \Rightarrow f(x_1) > f(x_2)$.

Thus, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.
This shows that $f(x)$ is decreasing on $]a, b[$.

SUMMARY

(i) If $f'(x) > 0$ for all $x \in]a, b[$ then $f(x)$ is increasing on $]a, b[$.

(ii) If $f'(x) < 0$ for all $x \in]a, b[$ then $f(x)$ is decreasing on $]a, b[$.

COROLLARY 1 *If $f'(x) > 0$ for all $x \in]a, b[$, except for a finite number of points where $f'(x) = 0$ then the function is increasing on $]a, b[$.*

PROOF Let $f'(x) > 0$ for all $x \in]a, b[$ except at one point c, where $f'(c) = 0$.
Then, $f'(x) > 0$ for all $x \in]a, c[$ and $f'(x) > 0$ for all $x \in]c, b[$.
Thus, $f(x)$ is increasing on $]a, c[$ as well as on $]c, b[$.

Hence, $f(x)$ is increasing on $]a, b[$.

The result may now be extended to the case when $f'(x) = 0$ at a finite number of points.

COROLLARY 2 *If $f'(x) < 0$ for all $x \in]a, b[$, except for a finite number of points where $f'(x) = 0$, then $f(x)$ is decreasing on $]a, b[$.*

PROOF It is similar to that of Corollary 1.

Two Important Results

I. (i) If $f'(x) \geq 0$ for all $x \in]a, b[$ then $f(x)$ is increasing on $[a, b]$.

(ii) If $f'(x) > 0$ for all $x \in]a, b[$ then $f(x)$ is strictly increasing on $[a, b]$.

II. (i) If $f'(x) \leq 0$ for all $x \in]a, b[$ then $f(x)$ is decreasing on $[a, b]$.

(ii) If $f'(x) < 0$ for all $x \in]a, b[$ then $f(x)$ is strictly decreasing on $[a, b]$.

SOLVED EXAMPLES

EXAMPLE 1 *Show that the function $f(x) = (x^3 - 6x^2 + 12x - 18)$ is an increasing function on R.* **[CBSE 2002C]**

SOLUTION $f(x) = (x^3 - 6x^2 + 12x - 18)$

$\Rightarrow \quad f'(x) = 3x^2 - 12x + 12$

$= 3(x^2 - 4x + 4) = 3(x-2)^2 \geq 0$ for all $x \in R$.

Thus, $f'(x) \geq 0$ for all $x \in R$.

Hence, $f(x)$ is an increasing function on R.

EXAMPLE 2 *Show that the function $f(x) = e^x$ is strictly increasing on R.*

SOLUTION $f(x) = e^x \Rightarrow f'(x) = e^x$.

Case I *When $x > 0$*

In this case, $e^x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) > 1$

$\Rightarrow \quad e^x > 0$ for all $x > 0$

$\Rightarrow \quad f'(x) > 0$ for all $x > 0$.

Case II *When $x = 0$*

In this case, $e^x = e^0 = 1 > 0$

$\Rightarrow \quad f'(x) > 0$ when $x = 0$.

Case III *When $x < 0$*

Let $x = -y$, where y is positive.

$$\text{Then,} \quad e^x = e^{-y} = \frac{1}{e^y} = \frac{1}{(\text{a } +\text{ve quantity})} > 0.$$

So, $f'(x) > 0$ when $x < 0$.

Thus, from all the cases, we have $f'(x) > 0$ for all $x \in R$.

Hence, $f(x)$ is strictly increasing for all $x \in R$.

EXAMPLE 3 *Show that $f(x) = e^{1/x}$ is a strictly decreasing function for all $x > 0$.*

SOLUTION $f(x) = e^{1/x} \Rightarrow f'(x) = -\frac{1}{x^2} e^{1/x} < 0$ for all $x > 0$

$[\because \ x^2 > 0$ and $e^{1/x} > 0$ when $x > 0]$.

Thus, $f'(x) < 0$ for all $x > 0$.

Hence, $f(x)$ is strictly decreasing for all $x > 0$.

EXAMPLE 4 *Show that $f(x) = (x-1)\, e^x + 1$ is a strictly increasing function for all $x > 0$.*

SOLUTION $f(x) = (x-1)\, e^x + 1 \Rightarrow f'(x) = (x-1)\, e^x + e^x$

$\Rightarrow f'(x) = x\, e^x > 0$ for all $x > 0$

$[\because \ x > 0$ and $e^x > 0$ when $x > 0]$.

Thus, $f'(x) > 0$ for all $x > 0$.

Hence, $f(x)$ is a strictly increasing function for all $x > 0$.

EXAMPLE 5 *Show that the function $f(x) = (x - \sin x)$ is increasing for all $x \in R$.*

SOLUTION $f(x) = (x - \sin x)$

$\Rightarrow \ f'(x) = (1 - \cos x) \geq 0$ for all $x \in R \quad [\because \ -1 \leq \cos x \leq 1]$

$\Rightarrow \ f'(x) \geq 0$ for all $x \in R$.

Hence, $f(x) = (x - \sin x)$ is increasing for all $x \in R$.

EXAMPLE 6 *Show that the function $f(x) = \cos^2 x$ is strictly decreasing on $\left]0, \frac{\pi}{2}\right[$.*

SOLUTION $f(x) = \cos^2 x$

$\Rightarrow \ f'(x) = -2\cos x \sin x = -(\sin 2x) < 0$ in $\left]0, \frac{\pi}{2}\right[$

$[\because \ \sin 2x > 0$ in the 1st quadrant$]$

$\Rightarrow \ f'(x) < 0$ for all $x \in \left]0, \frac{\pi}{2}\right[$.

Hence, $f(x) = \cos^2 x$ is strictly decreasing on $\left]0, \frac{\pi}{2}\right[$.

EXAMPLE 7 *Show that $f(x) = \log \sin x$ is (a) strictly increasing on $\left]0, \frac{\pi}{2}\right[$ and (b) strictly decreasing on $\left]\frac{\pi}{2}, \pi\right[$.*

SOLUTION $f(x) = \log \sin x$

$\Rightarrow \quad f'(x) = \dfrac{1}{\sin x} \cdot \cos x \Rightarrow f'(x) = \cot x \qquad \ldots(i)$

(a) We know that for each $x \in \left]0, \dfrac{\pi}{2}\right[$,

$f'(x) = \cot x > 0.$

$\therefore \quad f(x)$ is strictly increasing in $\left]0, \dfrac{\pi}{2}\right[$.

(b) We know that for each $x \in \left]\dfrac{\pi}{2}, \pi\right[$,

$f'(x) = \cot x < 0.$

$\therefore \quad f(x)$ is strictly decreasing in $\left]\dfrac{\pi}{2}, \pi\right[$.

EXAMPLE 8 *Show that the function* $f(x) = \sin x$ *is*

(a) strictly increasing on $\left]0, \dfrac{\pi}{2}\right[$

(b) strictly decreasing on $\left]\dfrac{\pi}{2}, \pi\right[$

(c) neither increasing nor decreasing on $]0, \pi[$

SOLUTION $f(x) = \sin x \Rightarrow f'(x) = \cos x.$

(a) We know that for each $x \in \left]0, \dfrac{\pi}{2}\right[$, $\cos x > 0.$

$\therefore \quad f'(x) > 0$ for all $x \in \left]0, \dfrac{\pi}{2}\right[$.

Hence, $f(x)$ is increasing on $\left]0, \dfrac{\pi}{2}\right[$.

(b) We know that for each $x \in \left]\dfrac{\pi}{2}, \pi\right[$, $\cos x < 0.$

$\therefore \quad f'(x) < 0$ for all $x \in \left]\dfrac{\pi}{2}, \pi\right[$.

Hence, $f(x)$ is decreasing on $\left]\dfrac{\pi}{2}, \pi\right[$.

(c) It follows from the above two results that $f(x) = \sin x$ is increasing on $\left]0, \dfrac{\pi}{2}\right[$ and decreasing on $\left]\dfrac{\pi}{2}, \pi\right[$.

So, it is neither increasing nor decreasing on $]0, \pi[$.

EXAMPLE 9 *Show that the function* $f(x) = (x^2 - x + 1)$ *is neither increasing nor decreasing on* $]0, 1[$.

SOLUTION $f(x) = (x^2 - x + 1) \Rightarrow f'(x) = (2x - 1).$

Now, $f'(x) > 0 \Leftrightarrow (2x-1) > 0 \Leftrightarrow x > \frac{1}{2}$.

$\therefore$ $f(x)$ is increasing when $x \in \left]\frac{1}{2}, 1\right[$.

And, $f'(x) < 0 \Leftrightarrow (2x-1) < 0 \Leftrightarrow x < \frac{1}{2}$.

$\therefore$ $f(x)$ is decreasing when $x \in \left]0, \frac{1}{2}\right[$.

Hence, the given function is neither increasing nor decreasing on $]0, 1[$.

EXAMPLE 10 *Prove that the function $f(x) = 10^x$ is strictly increasing on R.*

SOLUTION $f(x) = 10^x \Rightarrow f'(x) = (10^x)(\log 10) > 0$ for all $x \in R$.

Hence, $f(x) = 10^x$ is strictly increasing on R.

EXAMPLE 11 *Show that $f(x) = \tan^{-1}(\cos x + \sin x)$ is a strictly increasing function on the interval $\left(0, \frac{\pi}{4}\right)$.* **[CBSE 2007]**

SOLUTION $f(x) = \tan^{-1}(\cos x + \sin x)$

$$\Rightarrow f'(x) = \frac{1}{1+(\cos x + \sin x)^2} \cdot \frac{d}{dx}(\cos x + \sin x)$$

$$= \frac{(-\sin x + \cos x)}{(1 + \cos^2 x + \sin^2 x + 2\sin x \cos x)}$$

$$= \frac{(\cos x - \sin x)}{(2 + \sin 2x)}$$

Now, when $0 < x < \frac{\pi}{4}$, we have $\cos x > \sin x$ and $\sin 2x > 0$.

$\therefore$ $(\cos x - \sin x) > 0$ and $(2 + \sin 2x) > 0$.

$\therefore$ $f'(x) > 0$ for all x when $0 < x < \frac{\pi}{4}$.

Hence, $f(x)$ is strictly increasing in $\left(0, \frac{\pi}{4}\right)$.

EXAMPLE 12 *Find the intervals on which the function $f(x) = 10 - 6x - 2x^2$ is*
(a) strictly increasing (b) strictly decreasing.

SOLUTION $f(x) = 10 - 6x - 2x^2$

$$\Rightarrow f'(x) = -6 - 4x = -2(3 + 2x) = -4\left(x + \frac{3}{2}\right) \quad \ldots(i)$$

(a) $f(x)$ is strictly increasing

$\Leftrightarrow f'(x) > 0$

$\Leftrightarrow -4\left(x + \frac{3}{2}\right) > 0$ [from (i)]

$$\Leftrightarrow \quad \left(x+\frac{3}{2}\right)<0 \quad \Leftrightarrow \quad x<\frac{-3}{2}.$$

$\therefore$ $f(x)$ is strictly increasing on the interval $\left]-\infty, \frac{-3}{2}\right[$.

(b) $f(x)$ is strictly decreasing

$$\Leftrightarrow \quad f'(x)<0$$
$$\Leftrightarrow \quad -4\left(x+\frac{3}{2}\right)<0 \qquad \text{[from (i)]}$$
$$\Leftrightarrow \quad \left(x+\frac{3}{2}\right)>0$$
$$\Leftrightarrow \quad x>\frac{-3}{2}.$$

$\therefore$ $f(x)$ is strictly decreasing on the interval $\left]\frac{-3}{2}, \infty\right[$.

Hence, $f(x)$ is strictly increasing on the interval $\left]-\infty, \frac{-3}{2}\right[$ and strictly decreasing on $\left]\frac{-3}{2}, \infty\right[$.

EXAMPLE 13 *Find the intervals on which the function* $f(x)=-2x^3-9x^2-12x+1$ *is (a) strictly increasing (b) strictly decreasing.*

SOLUTION $f(x)=-2x^3-9x^2-12x+1$

$$\Rightarrow f'(x)=-6x^2-18x-12=-6(x^2+3x+2)=-6(x+2)(x+1) \quad \ldots(\text{i})$$

(a) $f(x)$ is strictly increasing

$$\Leftrightarrow \quad f'(x)>0$$
$$\Leftrightarrow \quad -6(x+2)(x+1)>0 \qquad \text{[from (i)]}$$
$$\Leftrightarrow \quad (x+2)(x+1)<0$$
$$\Leftrightarrow \quad -2<x<-1$$
$$\Leftrightarrow \quad x\in\,]-2,-1[.$$

$\therefore$ $f(x)$ is strictly increasing on the interval $]-2,-1[$.

(b) $f(x)$ is strictly decreasing

$\Leftrightarrow$ $f'(x)<0$

$\Leftrightarrow$ $-6(x+2)(x+1)<0$ [from (i)]

$\Leftrightarrow$ $(x+2)(x+1)>0$

$\Leftrightarrow$ $[(x+2)>0$ and $(x+1)>0]$
or $[(x+2)<0$ and $(x+1)<0]$

$\Leftrightarrow$ $(x>-2$ and $x>-1)$ or $(x<-2$ and $x<-1)$

$\Leftrightarrow$ $(x>-1)$ or $(x<-2)$

$\Leftrightarrow$ $x\in\,]-1,\infty[$ or $x\in\,]-\infty,-2[$

$\Leftrightarrow$ $x\in\,]-\infty,-2[\cup\,]-1,\infty[$.

$\therefore$ $f(x)$ is strictly decreasing on $]-\infty,-2[\cup\,]-1,\infty[$.

EXAMPLE 14 *Find the intervals on which the function* $f(x) = 2x^3 - 15x^2 + 36x + 6$ *is (a) increasing (b) decreasing.* **[CBSE 2004C, '05, '09C]**

SOLUTION $f(x) = 2x^3 - 15x^2 + 36x + 6$

$\Rightarrow \quad f'(x) = 6x^2 - 30x + 36 = 6(x^2 - 5x + 6) = 6(x-3)(x-2) \quad \ldots(i)$

(a) $f(x)$ is increasing

$\Leftrightarrow \quad f'(x) \geq 0$

$\Leftrightarrow \quad 6(x-3)(x-2) \geq 0 \qquad$ [from (i)]

$\Leftrightarrow \quad (x-3)(x-2) \geq 0$

$\Leftrightarrow \quad [(x-3) \geq 0 \text{ and } (x-2) \geq 0]$

or $\quad [(x-3) \leq 0 \text{ and } (x-2) \leq 0]$

$\Leftrightarrow \quad [x \geq 3 \text{ and } x \geq 2] \text{ or } [x \leq 3 \text{ and } x \leq 2]$

$\Leftrightarrow \quad [x \geq 3] \text{ or } [x \leq 2]$

$\Leftrightarrow \quad x \in [3, \infty[\text{ or } x \in]-\infty, 2]$

$\Leftrightarrow \quad x \in]-\infty, 2] \cup [3, \infty].$

$\therefore \quad f(x)$ is increasing on $]-\infty, 2] \cup [3, \infty[.$

(b) $f(x)$ is decreasing

$\Leftrightarrow \quad f'(x) \leq 0$

$\Leftrightarrow \quad 6(x-3)(x-2) \leq 0 \qquad$ [from (i)]

$\Leftrightarrow \quad (x-3)(x-2) \leq 0$

$\Leftrightarrow \quad 2 \leq x \leq 3$

$\Leftrightarrow \quad x \in [2, 3].$

$\therefore \quad f(x)$ is decreasing on $[2, 3]$.

Hence, $f(x)$ is increasing on $]-\infty, 2] \cup [3, \infty[$ and decreasing on $[2, 3]$.

EXAMPLE 15 *Find the intervals on which the function* $f(x) = x^3 + 2x^2 - 1$ *is (a) increasing (b) decreasing.*

SOLUTION $f(x) = x^3 + 2x^2 - 1$

$\Rightarrow \quad f'(x) = 3x^2 + 4x = 3x\left(x + \frac{4}{3}\right) \quad \ldots(i)$

(a) $f(x)$ is increasing

$\Leftrightarrow \quad f'(x) \geq 0$

$\Leftrightarrow \quad 3x\left(x + \frac{4}{3}\right) \geq 0 \qquad$ [from (i)]

$\Leftrightarrow \quad x\left(x + \frac{4}{3}\right) \geq 0$

$\Leftrightarrow \quad [x \geq 0 \text{ and } \left(x + \frac{4}{3}\right) \geq 0] \text{ or } [x \leq 0 \text{ and } \left(x + \frac{4}{3}\right) \leq 0]$

$\Leftrightarrow \quad [x \geq 0 \text{ and } x \geq -\frac{4}{3}] \text{ or } [x \leq 0 \text{ and } x \leq -\frac{4}{3}]$

$\Leftrightarrow \quad (x \geq 0) \text{ or } \left(x \geq -\frac{4}{3}\right)$

$\Leftrightarrow \quad x \in [0, \infty[$ or $x \in]-\infty, -\frac{4}{3}]$

$\Leftrightarrow \quad x \in \left]-\infty, -\frac{4}{3}\right] \cup [0, \infty[.$

$\therefore \quad f(x)$ is increasing on $\left]-\infty, -\frac{4}{3}\right] \cup [0, \infty[.$

(b) $f(x)$ is decreasing

$\Leftrightarrow \quad f'(x) \le 0$

$\Leftrightarrow \quad 3x\left(x + \frac{4}{3}\right) \le 0 \qquad$ [from (i)]

$\Leftrightarrow \quad x\left(x + \frac{4}{3}\right) \le 0$

$\Leftrightarrow \quad -\frac{4}{3} \le x \le 0$

$\Leftrightarrow \quad x \in \left[-\frac{4}{3}, 0\right].$

$\therefore \quad f(x)$ is decreasing on $\left[-\frac{4}{3}, 0\right].$

Hence, $f(x)$ is increasing on $\left]-\infty, -\frac{4}{3}\right] \cup [0, \infty[$ and decreasing on $\left[-\frac{4}{3}, 0\right].$

EXAMPLE 16 *Find the intervals on which the function* $f(x) = x^3 + 3x^2 - 105x + 25$ *is (a) increasing (b) decreasing.*

SOLUTION $f(x) = x^3 + 3x^2 - 105x + 25$

$\Rightarrow \quad f'(x) = 3x^2 + 6x - 105 = 3(x^2 + 2x - 35) = 3(x+7)(x-5) \quad \ldots(i)$

(a) $f(x)$ is increasing

$\Leftrightarrow \quad f'(x) \ge 0$

$\Leftrightarrow \quad 3(x+7)(x-5) \ge 0 \qquad$ [from (i)]

$\Leftrightarrow \quad (x+7)(x-5) \ge 0$

$\Leftrightarrow \quad [(x+7) \ge 0 \text{ and } (x-5) \ge 0]$ or $[(x+7) \le 0 \text{ and } (x-5) \le 0]$

$\Leftrightarrow \quad [x \ge -7$ and $x \ge 5]$ or $[x \le -7$ and $x \le 5]$

$\Leftrightarrow \quad (x \ge 5)$ or $(x \le -7)$

$\Leftrightarrow \quad x \in [5, \infty[$ or $x \in]-\infty, -7]$

$\Leftrightarrow \quad x \in]-\infty, -7] \cup [5, \infty[.$

$\therefore \quad f(x)$ is increasing on $]-\infty, -7] \cup [5, \infty[$.

(b) $f(x)$ is decreasing

$\Leftrightarrow \quad f'(x) \le 0$

$\Leftrightarrow \quad 3(x+7)(x-5) \le 0 \qquad$ [from (i)]

$\Leftrightarrow \quad (x+7)(x-5) \le 0$

$\Leftrightarrow \quad -7 \le x \le 5$

$\Leftrightarrow \quad x \in [-7, 5].$

$\therefore$ $f(x)$ is decreasing on $[-7, 5]$.

Hence, $f(x)$ is increasing on $]-\infty, -7] \cup [5, \infty[$ and decreasing on $[-7, 5]$.

EXAMPLE 17 *Find the intervals on which the function* $f(x) = 5 + 36x + 3x^2 - 2x^3$ *is (a) increasing (b) decreasing.*

SOLUTION $f(x) = 5 + 36x + 3x^2 - 2x^3$

$\Rightarrow$ $f'(x) = 36 + 6x - 6x^2 = -6(x^2 - x - 6) = -6(x + 2)(x - 3)$ ...(i)

(a) $f(x)$ is increasing

$\Leftrightarrow$ $f'(x) \geq 0$

$\Leftrightarrow$ $-6(x + 2)(x - 3) \geq 0$ [from (i)]

$\Leftrightarrow$ $(x + 2)(x - 3) \leq 0$

$\Leftrightarrow$ $-2 \leq x \leq 3$

$\Leftrightarrow$ $x \in [-2, 3]$.

$\therefore$ $f(x)$ is increasing on $[-2, 3]$.

(b) $f(x)$ is decreasing

$\Leftrightarrow$ $f'(x) \leq 0$

$\Leftrightarrow$ $-6(x + 2)(x - 3) \leq 0$ [from (i)]

$\Leftrightarrow$ $(x + 2)(x - 3) \leq 0$

$\Leftrightarrow$ $[(x + 2) \geq 0 \text{ and } (x - 3) \geq 0]$ or $[(x + 2) \leq 0 \text{ and } (x - 3) \leq 0]$

$\Leftrightarrow$ $[x \geq -2 \text{ and } x \geq 3]$ or $[x \leq -2 \text{ and } x \leq 3]$

$\Leftrightarrow$ $(x \geq 3)$ or $(x \leq -2)$

$\Leftrightarrow$ $x \in [3, \infty[$ or $x \in]-\infty, -2]$

$\Leftrightarrow$ $x \in]-\infty, -2] \cup [3, \infty[$.

$\therefore$ $f(x)$ is decreasing on $]-\infty, -2] \cup [3, \infty[$.

Hence, $f(x)$ is increasing on $[-2, 3]$ and decreasing on $]-\infty, -2] \cup [3, \infty[$.

EXAMPLE 18 *Find the intervals on which the function* $f(x) = (x + 1)^3(x - 3)^3$ *is (a) increasing (b) decreasing.* **[CBSE 2001C]**

SOLUTION $f(x) = (x + 1)^3(x - 3)^3$

$\Rightarrow$ $f'(x) = (x + 1)^3 \cdot \frac{d}{dx}(x - 3)^3 + (x - 3)^3 \cdot \frac{d}{dx}(x + 1)^3$

$= (x + 1)^3 \cdot 3(x - 3)^2 + (x - 3)^3 \cdot 3(x + 1)^2$

$= 3(x + 1)^2(x - 3)^2[(x + 1) + (x - 3)]$

$= 6(x + 1)^2(x - 3)^2(x - 1)$ ...(i)

(a) $f(x)$ is increasing

$\Leftrightarrow$ $f'(x) \geq 0$

$\Leftrightarrow$ $6(x + 1)^2(x - 3)^2(x - 1) \geq 0$ [from (i)]

$\Leftrightarrow$ $(x - 1) \geq 0$

$\Leftrightarrow$ $x \geq 1$

$\Leftrightarrow$ $x \in [1, \infty[$.

$\therefore$ $f(x)$ is increasing on $[1, \infty[$.

(b) $f(x)$ is decreasing

$\Leftrightarrow \quad f'(x) \le 0$

$\Leftrightarrow \quad 6(x+1)^2(x-3)^2(x-1) \le 0 \qquad$ [from (i)]

$\Leftrightarrow \quad (x-1) \le 0 \quad \Leftrightarrow \quad x \le 1$

$\Leftrightarrow \quad x \in]-\infty, 1]$.

$\therefore \quad f(x)$ is decreasing on $]-\infty, 1]$.

Hence, $f(x)$ is increasing on $[1, \infty[$ and decreasing on $]-\infty, 1]$.

EXAMPLE 19 *Find the intervals on which the function* $f(x) = \dfrac{4x^2+1}{x}, (x \ne 0)$ *is (a) increasing (b) decreasing.* **[CBSE 2004]**

SOLUTION $\quad f(x) = \dfrac{4x^2+1}{x}, x \ne 0$

$\Rightarrow \quad f(x) = \left(4x + \dfrac{1}{x}\right), x \ne 0$

$\Rightarrow \quad f'(x) = \left(4 - \dfrac{1}{x^2}\right)$

$\Rightarrow \quad f'(x) = \dfrac{(4x^2-1)}{x^2} \qquad \ldots(i)$

(a) $f(x)$ is increasing

$\Leftrightarrow \quad f'(x) \ge 0 \Leftrightarrow \dfrac{(4x^2-1)}{x^2} \ge 0 \qquad$ [from (i)]

$\Leftrightarrow \quad (4x^2-1) \ge 0 \qquad [\because\ x^2 > 0]$

$\Leftrightarrow \quad (2x-1)(2x+1) \ge 0 \Leftrightarrow 2\left(x - \dfrac{1}{2}\right) \cdot 2\left(x + \dfrac{1}{2}\right) \ge 0$

$\Leftrightarrow \quad \left(x - \dfrac{1}{2}\right)\left(x + \dfrac{1}{2}\right) \ge 0$

$\Leftrightarrow \quad \left[\left(x - \dfrac{1}{2}\right) \ge 0 \text{ and } \left(x + \dfrac{1}{2}\right) \ge 0\right]$

or $\quad \left[\left(x - \dfrac{1}{2}\right) \le 0 \text{ and } \left(x + \dfrac{1}{2}\right) \le 0\right]$

$\Leftrightarrow \quad \left[x \ge \dfrac{1}{2} \text{ and } x \ge -\dfrac{1}{2}\right] \text{ or } \left[x \le \dfrac{1}{2} \text{ and } x \le -\dfrac{1}{2}\right]$

$\Leftrightarrow \quad \left(x \ge \dfrac{1}{2}\right) \text{ or } \left(x \le -\dfrac{1}{2}\right)$

$\Leftrightarrow \quad x \in \left[\dfrac{1}{2}, \infty\right[\text{ or } x \in \left]-\infty, -\dfrac{1}{2}\right]$

$\Leftrightarrow \quad x \in \left]-\infty, -\dfrac{1}{2}\right] \cup \left[\dfrac{1}{2}, \infty\right[$.

$\therefore \quad f(x)$ is increasing on $\left]-\infty, -\dfrac{1}{2}\right] \cup \left[\dfrac{1}{2}, \infty\right[$.

(b) $f(x)$ is decreasing

$\Leftrightarrow \quad f'(x) \le 0 \Leftrightarrow \dfrac{(4x^2-1)}{x^2} \le 0 \qquad$ [from (i)]

$\Leftrightarrow \quad (4x^2-1) \le 0 \qquad [\because\ x^2 > 0]$

$\Leftrightarrow \quad (2x-1)(2x+1) \le 0 \Leftrightarrow 2\left(x-\dfrac{1}{2}\right)\cdot 2\left(x+\dfrac{1}{2}\right) \le 0$

$\Leftrightarrow \quad \left(x-\dfrac{1}{2}\right)\left(x+\dfrac{1}{2}\right) \le 0$

$\Leftrightarrow \quad -\dfrac{1}{2} \le x \le \dfrac{1}{2}$

$\Leftrightarrow \quad x \in \left[-\dfrac{1}{2}, \dfrac{1}{2}\right].$

$\therefore\ f(x)$ is decreasing on $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right].$

Hence, $f(x)$ is increasing on $\left]-\infty, -\dfrac{1}{2}\right] \cup \left[\dfrac{1}{2}, \infty\right[$ and decreasing on $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right].$

EXAMPLE 20 *Find the intervals on which the function* $f(x) = \dfrac{x}{(x^2+1)}$ *is*

(a) increasing (b) decreasing. **[CBSE 2003]**

SOLUTION $f(x) = \dfrac{x}{(x^2+1)}$

$\Rightarrow \quad f'(x) = \dfrac{(x^2+1)\cdot \dfrac{d}{dx}(x) - x\cdot \dfrac{d}{dx}(x^2+1)}{(x^2+1)^2} = \dfrac{(x^2+1)\cdot 1 - x\cdot 2x}{(x^2+1)^2}$

$\Rightarrow \quad f'(x) = \dfrac{(1-x^2)}{(x^2+1)^2} \qquad \ldots(i)$

(a) $f(x)$ is increasing

$\Leftrightarrow \quad f'(x) \ge 0$

$\Leftrightarrow \quad \dfrac{(1-x^2)}{(x^2+1)^2} \ge 0 \qquad$ [from (i)]

$\Leftrightarrow \quad (1-x^2) \ge 0 \qquad [\because\ (x^2+1)^2 > 0]$

$\Leftrightarrow \quad -(1-x^2) \le 0 \Leftrightarrow (x^2-1) \le 0$

$\Leftrightarrow \quad (x-1)(x+1) \le 0$

$\Leftrightarrow \quad -1 \le x \le 1$

$\Leftrightarrow \quad x \in [-1, 1].$

$\therefore\ f(x)$ is increasing on $[-1, 1].$

(b) $f(x)$ is decreasing

$\Leftrightarrow \quad f'(x) \le 0$

$\Leftrightarrow \quad \dfrac{(1-x^2)}{(x^2+1)^2} \le 0$ [from (i)]

$\Leftrightarrow \quad (1-x^2) \le 0 \qquad [\because\ (x^2+1)^2 > 0]$

$\Leftrightarrow \quad -(1-x^2) \ge 0$

$\Leftrightarrow \quad (x^2-1) \le 0 \quad \Leftrightarrow \quad (x-1)(x+1) \ge 0$

$\Leftrightarrow \quad [(x-1) \ge 0 \text{ and } (x+1) \ge 0] \text{ or } [(x-1) \le 0 \text{ and } (x+1) \le 0]$

$\Leftrightarrow \quad [x \ge 1 \text{ and } x \ge -1] \text{ or } [x \le 1 \text{ and } x \le -1]$

$\Leftrightarrow \quad (x \ge 1) \text{ or } (x \le -1)$

$\Leftrightarrow \quad x \in [1, \infty[\text{ or } x \in]-\infty, -1]$

$\Leftrightarrow \quad x \in]-\infty, -1] \cup [1, \infty[.$

$\therefore \quad f(x)$ is decreasing on $]-\infty, -1] \cup [1, \infty[.$

Hence, $f(x)$ is increasing on $[-1, 1]$ and decreasing on $]-\infty, -1] \cup [1, \infty[$.

EXAMPLE 21 *Find the intervals on which the function* $f(x) = (x+2)\ e^{-x}$ *is*

(a) increasing (b) decreasing. **[CBSE 2000C]**

SOLUTION $f(x) = (x+2)\ e^{-x}$

$\Rightarrow \quad f'(x) = -(x+2)e^{-x} + e^{-x} \cdot 1$

$\Rightarrow \quad f'(x) = -(x+1)e^{-x} \qquad \ldots(i)$

(a) $f(x)$ is increasing

$\Leftrightarrow \quad f'(x) \ge 0$

$\Leftrightarrow \quad -(x+1)e^{-x} \ge 0$ [from (i)]

$\Leftrightarrow \quad (x+1)e^{-x} \le 0$

$\Leftrightarrow \quad (x+1) \le 0 \qquad [\because\ e^{-x} > 0]$

$\Leftrightarrow \quad x \le -1 \quad \Leftrightarrow \quad x \in]-\infty, -1].$

$\therefore \quad f(x)$ is increasing on $]-\infty, -1].$

(b) $f(x)$ is decreasing

$\Leftrightarrow \quad f'(x) \le 0$

$\Leftrightarrow \quad -(x+1)e^{-x} \le 0$ [from (i)]

$\Leftrightarrow \quad (x+1)e^{-x} \ge 0$

$\Leftrightarrow \quad (x+1) \ge 0 \qquad [\because\ e^{-x} > 0]$

$\Leftrightarrow \quad x \ge -1$

$\Leftrightarrow \quad x \in [-1, \infty[.$

$\therefore \quad f(x)$ is decreasing on $[-1, \infty[.$

Hence, $f(x)$ is increasing on $]-\infty, -1]$ and decreasing on $[-1, \infty[.$

EXAMPLE 22 *Find the intervals on which the function* $f(x) = \log(1+x) - \dfrac{x}{(1+x)}$ *is*

(a) increasing (b) decreasing. **[CBSE 2000C]**

SOLUTION $f(x) = \log(1+x) - \dfrac{x}{(1+x)}$

$$\Rightarrow \quad f'(x) = \left\{\frac{1}{(1+x)} - \frac{(1+x)\cdot 1 - x\cdot 1}{(1+x)^2}\right\}$$

$$= \left\{\frac{1}{(1+x)} - \frac{1}{(1+x)^2}\right\} = \frac{(1+x-1)}{(1+x)^2}$$

$$\Rightarrow \quad f'(x) = \frac{x}{(1+x)^2} \qquad \ldots(\text{i})$$

(a) $f(x)$ is increasing

$\Leftrightarrow \quad f'(x) \geq 0$

$\Leftrightarrow \quad \frac{x}{(1+x)^2} \geq 0$ [from (i)]

$\Leftrightarrow \quad x \geq 0$ [$\because (1+x)^2 \geq 0$]

$\Leftrightarrow \quad x \in [0, \infty[$.

(b) $f(x)$ is decreasing

$\Leftrightarrow \quad f'(x) \leq 0$

$\Leftrightarrow \quad \frac{x}{(1+x)^2} \leq 0$ [from (i)]

$\Leftrightarrow \quad x \leq 0$

$\Leftrightarrow \quad x \in]-\infty, 0]$.

Hence, $f(x)$ is increasing on $[0, \infty[$ and decreasing on $]-\infty, 0]$.

EXAMPLE 23 *Find the intervals on which the function* $f(x) = (\sin x - \cos x)$, $0 < x < 2\pi$, *is (a) increasing (b) decreasing.* **[CBSE 2000C, '09, '11]**

SOLUTION $f(x) = (\sin x - \cos x)$, $0 < x < 2\pi$

$$\Rightarrow \quad f'(x) = (\cos x + \sin x) = \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x\right)$$

$$\Rightarrow \quad f'(x) = \sqrt{2}\left(\sin\frac{\pi}{4}\cos x + \cos\frac{\pi}{4}\sin x\right)$$

$$\Rightarrow \quad f'(x) = \sqrt{2}\sin\left(\frac{\pi}{4} + x\right) \qquad \ldots(\text{i})$$

Now, $0 < x < 2\pi \Rightarrow \frac{\pi}{4} < \left(\frac{\pi}{4} + x\right) < \left(\frac{\pi}{4} + 2\pi\right)$

$$\Rightarrow \quad \frac{\pi}{4} < \left(\frac{\pi}{4} + x\right) < \frac{9\pi}{4} \qquad \ldots(\text{ii})$$

(a) $f(x)$ is increasing

$\Leftrightarrow \quad f'(x) \geq 0$

$\Leftrightarrow \quad \sqrt{2}\sin\left(\frac{\pi}{4} + x\right) \geq 0$ [from (i)]

$\Leftrightarrow \quad \sin\left(\frac{\pi}{4} + x\right) \geq 0$

$$\Leftrightarrow \quad \left\{\frac{\pi}{4} < \left(\frac{\pi}{4} + x\right) \le \pi\right\} \text{ or } \left\{2\pi \le \left(\frac{\pi}{4} + x\right) < \frac{9\pi}{4}\right\}$$

$$\Leftrightarrow \quad \left\{0 < x \le \frac{3\pi}{4}\right\} \text{ or } \left\{\frac{7\pi}{4} \le x < 2\pi\right\}$$

$$\Leftrightarrow \quad x \in \left]0, \frac{3\pi}{4}\right] \text{ or } x \in \left[\frac{7\pi}{4}, 2\pi\right[$$

$$\Leftrightarrow \quad x \in \left]0, \frac{3\pi}{4}\right] \cup \left[\frac{7\pi}{4}, 2\pi\right[.$$

$\therefore f(x)$ is increasing on $\left]0, \frac{3\pi}{4}\right] \cup \left[\frac{7\pi}{4}, 2\pi\right[$.

(b) $f(x)$ is decreasing

$$\Leftrightarrow \quad f'(x) \le 0$$

$$\Leftrightarrow \quad \sqrt{2}\sin\left(\frac{\pi}{4} + x\right) \le 0 \qquad \text{[from (i)]}$$

$$\Leftrightarrow \quad \sin\left(\frac{\pi}{4} + x\right) \le 0$$

$$\Leftrightarrow \quad \pi \le \left(\frac{\pi}{4} + x\right) \le 2\pi$$

$$\Leftrightarrow \quad \frac{3\pi}{4} \le x \le \frac{7\pi}{4}$$

$$\Leftrightarrow \quad x \in \left[\frac{3\pi}{4}, \frac{7\pi}{4}\right].$$

$\therefore f(x)$ is decreasing on $\left[\frac{3\pi}{4}, \frac{7\pi}{4}\right]$.

EXAMPLE 24 *Separate the interval* $\left[0, \frac{\pi}{2}\right]$ *into subintervals in which* $f(x) = (\sin^4 x + \cos^4 x)$ *is (a) increasing (b) decreasing.* **[CBSE 2000C]**

SOLUTION $f(x) = (\sin^4 x + \cos^4 x)$

$$\Rightarrow \quad f'(x) = 4\sin^3 x \cos x - 4\cos^3 x \sin x$$

$$= -4\sin x \cos x(\cos^2 x - \sin^2 x)$$

$$= -2\sin 2x \cos 2x = -\sin 4x \qquad \ldots(i)$$

Also, $0 \le x \le \frac{\pi}{2} \Leftrightarrow 0 \le 4x \le 2\pi$.

(a) $f(x)$ is increasing

$$\Leftrightarrow \quad f'(x) \ge 0$$

$$\Leftrightarrow \quad -\sin 4x \ge 0 \qquad \text{[from (i)]}$$

$$\Leftrightarrow \quad \sin 4x \le 0$$

$$\Leftrightarrow \quad \pi \le 4x \le 2\pi$$

$$\Leftrightarrow \quad \frac{\pi}{4} \le x \le \frac{\pi}{2}$$

$\Leftrightarrow \quad x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$.

$\therefore \quad f(x)$ is increasing on $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$.

(b) $f(x)$ is decreasing

$\Leftrightarrow \quad f'(x) \leq 0$ [from (i)]

$\Leftrightarrow \quad -\sin 4x \leq 0$

$\Leftrightarrow \quad \sin 4x \geq 0$

$\Leftrightarrow \quad 0 \leq 4x \leq \pi$

$\Leftrightarrow \quad 0 \leq x \leq \frac{\pi}{4}$

$\Leftrightarrow \quad x \in \left[0, \frac{\pi}{4}\right]$.

$\therefore \quad f(x)$ is decreasing on $\left[0, \frac{\pi}{4}\right]$.

Hence, $f(x)$ is increasing on $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ and decreasing on $\left[0, \frac{\pi}{4}\right]$.

EXAMPLE 25 *Separate* $\left[0, \frac{\pi}{2}\right]$ *into subintervals in which* $f(x) = \sin 3x$ *is*

(a) increasing (b) decreasing.

SOLUTION $f(x) = \sin 3x \Rightarrow f'(x) = 3\cos 3x$ ...(i)

Also, $0 \leq x \leq \frac{\pi}{2} \Leftrightarrow 0 \leq 3x \leq \frac{3\pi}{2}$ ...(ii)

(a) $f(x)$ is increasing

$\Leftrightarrow \quad f'(x) \geq 0$

$\Leftrightarrow \quad 3\cos 3x \geq 0 \Leftrightarrow \cos 3x \geq 0$ [from (i)]

$\Leftrightarrow \quad 0 \leq 3x \leq \frac{\pi}{2}$

$\Leftrightarrow \quad 0 \leq x \leq \frac{\pi}{6}$

$\Leftrightarrow \quad x \in \left[0, \frac{\pi}{6}\right]$.

$\therefore \quad f(x)$ is increasing on $\left[0, \frac{\pi}{6}\right]$.

(b) $f(x)$ is decreasing

$\Leftrightarrow \quad f'(x) \leq 0$

$\Leftrightarrow \quad 3\cos 3x \leq 0 \Leftrightarrow \cos 3x \geq 0$ [from (i)]

$\Leftrightarrow \quad \frac{\pi}{2} \leq 3x \leq \frac{3\pi}{2}$

$\Leftrightarrow \quad \frac{\pi}{6} \leq x \leq \frac{\pi}{2}$

$\Leftrightarrow \quad x \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right]$.

$\therefore$ $f(x)$ is decreasing on $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$.

Hence, $f(x)$ is increasing on $\left[0, \frac{\pi}{6}\right]$ and decreasing on $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$.

EXAMPLE 26 *Prove that* $\tan x > x$ *for all* $x \in \left[0, \frac{\pi}{2}\right]$.

SOLUTION Let c be an arbitrary real number such that $c \in \left]0, \frac{\pi}{2}\right[$.

Let $f(x) = \tan x - x$ for all $x \in [0, c]$.

$\therefore$ $f'(x) = \sec^2 x - 1 = \tan^2 x > 0$ for all $x \in]0, c[$.

Thus, $f(x)$ is increasing on $]0, c[$.

Now, $x > 0 \Rightarrow f(x) > f(0)$

$\Rightarrow f(x) > 0$

$\Rightarrow \tan x - x > 0$

$\Rightarrow \tan x > x$.

EXAMPLE 27 *Find the values of x for which the function* $f(x) = x^x$, $x > 0$ *is*

(a) increasing (b) decreasing.

SOLUTION $f(x) = x^x$

$\Rightarrow f(x) = e^{x \log x}$

$\Rightarrow f'(x) = e^{x \log x} \cdot \frac{d}{dx}(x \log x)$

$\Rightarrow f'(x) = x^x(1 + \log_e x)$ …(i)

(a) $f(x)$ is increasing

$\Leftrightarrow f'(x) \geq 0$

$\Leftrightarrow x^x(1 + \log_e x) \geq 0$ [from (i)]

$\Leftrightarrow (1 + \log_e x) \geq 0$ [$\because x^x > 0$ when $x > 0$]

$\Leftrightarrow \log_e x \geq -1$

$\Leftrightarrow x \geq e^{-1}$

$\Leftrightarrow x \in \left[\frac{1}{e}, \infty\right[$.

$\therefore$ $f(x)$ is increasing on $\left[\frac{1}{e}, \infty\right[$.

(b) $f(x)$ is decreasing

$\Leftrightarrow f'(x) \leq 0$

$\Leftrightarrow x^x(1 + \log_e x) \leq 0$

$\Leftrightarrow (1 + \log_e x) \leq 0$ [$\because x^x > 0$]

$\Leftrightarrow \log_e x \leq -1$

$\Leftrightarrow x \leq e^{-1}$

$\Leftrightarrow 0 \leq x \leq \frac{1}{e}$

$\Leftrightarrow \quad x \in \left]0, \frac{1}{e}\right]$.

$\therefore \quad f(x)$ is decreasing on $\left]0, \frac{1}{e}\right]$.

Hence, $f(x)$ is increasing on $\left[\frac{1}{e}, \infty\right[$ and decreasing on $\left]0, \frac{1}{e}\right]$.

EXAMPLE 28 *Find the intervals on which the function* $f(x) = (x^4 - 2x^2)$ *is increasing or decreasing.*

SOLUTION $f(x) = x^4 - 2x^2 \Rightarrow f'(x) = 4x^3 - 4x = 4x(x-1)(x+1)$.

Now, $f'(x) = 0 \Rightarrow x = -1$ or $x = 0$ or $x = 1$.

These points divide the whole real line into four disjoint open intervals, namely, $]-\infty, -1[$, $]-1, 0[$, $]0, 1[$ and $]1, \infty[$.

Case I *When* $x \in]-\infty, -1[$

In this case, $x < -1$. So, $f'(x) = (-)(-)(-) = -\text{ve}$.

So, $f'(x) < 0$ for all $x \in]-\infty, -1[$.

$\therefore \quad f(x)$ is decreasing for all $x \in]-\infty, -1[$.

Case II *When* $x \in]-1, 0[$

In this case, $-1 < x < 0$.

So, $f'(x) = (-)(-)(+) = +\text{ve}$.

$\therefore \quad f'(x) > 0$ for all $x \in]-1, 0[$.

$\therefore \quad f(x)$ is increasing on $]-1, 0[$.

Case III *When* $x \in]0, 1[$

In this case, $0 < x < 1$.

So, $f'(x) = (+)(-)(+) = -\text{ve}$.

Thus, $f'(x) < 0$ for all $x \in]0, 1[$.

$\therefore \quad f(x)$ is decreasing on $]0, 1[$.

Case IV *When* $x \in]1, \infty[$

In this case, $1 < x < \infty$.

So, $f'(x) = (+)(+)(+) = +\text{ve}$.

Thus, $f'(x) > 0$ for all $x \in]1, \infty[$.

$\therefore \quad f(x)$ is increasing on $]1, \infty[$.

From all the four cases, we conclude that

$f(x)$ is increasing on $]-1, 0[\cup]1, \infty[$

and, $f(x)$ is decreasing on $]-\infty, -1[\cup]0, 1[$.

EXAMPLE 29 *Prove that* $\frac{x}{(1+x)} < \log(1+x) < x$ *for* $x > 0$.

SOLUTION Let $f(x) = \log(1+x) - \frac{x}{(1+x)}$.

Then, $f'(x) = \left[\frac{1}{1+x} - \frac{1}{(1+x)^2}\right] = \frac{x}{(1+x)^2} > 0$ for $x > 0$.

Since $f'(x) > 0$ for all $x > 0$ and $f'(0) = 0$, it follows that $f(x)$ is increasing on $[0, \infty[$.

Now, $x > 0 \Rightarrow f(x) > f(0) \Rightarrow f(x) > 0 \quad [\because f(0) = 0]$

$$\Rightarrow \left[\log(1+x) - \frac{x}{(1+x)}\right] > 0 \Rightarrow \log(1+x) > \frac{x}{(1+x)} \quad \ldots \text{(i)}$$

Again, let $g(x) = [x - \log(1+x)]$.

Then, $g'(x) = \left[1 - \frac{1}{(1+x)}\right] = \frac{x}{(1+x)} > 0$ for $x > 0$.

Now, $g'(x) > 0$ for all $x > 0$ and $g'(0) = 0$.

$\therefore$ $g(x)$ is strictly increasing on $[0, \infty[$.

Also, $g(0) = 0$.

Now, $x > 0 \Rightarrow g(x) > g(0) \Rightarrow g(x) > 0 \quad [\because g(0) = 0]$

$\Rightarrow [x - \log(1+x)] > 0 \Rightarrow x > \log(1+x) \quad \ldots \text{(ii)}$

From (i) and (ii), we get $\frac{x}{(1+x)} < \log(1+x) < x$ for $x > 0$.

EXERCISE 11G

1. Show that the function $f(x) = 5x - 2$ is a strictly increasing function on R.
2. Show that the function $f(x) = -2x + 7$ is a strictly decreasing function on R.
3. Prove that $f(x) = ax + b$, where a and b are constants and $a > 0$, is a strictly increasing function on R.
4. Prove that the function $f(x) = e^{2x}$ is strictly increasing on R.
5. Show that the function $f(x) = x^2$ is
   (a) strictly increasing on $[0, \infty[$
   (b) strictly decreasing on $]-\infty, 0[$
   (c) neither strictly increasing nor strictly decreasing on R
6. Show that the function $f(x) = |x|$ is
   (a) strictly increasing on $]0, \infty[$
   (b) strictly decreasing on $]-\infty, 0[$
7. Prove that the function $f(x) = \log_e x$ is strictly increasing on $]0, \infty[$.
8. Prove that the function $f(x) = \log_a x$ is strictly increasing on $]0, \infty[$ when $a > 1$ and strictly decreasing on $]0, \infty[$ when $0 < a < 1$.
9. Prove that $f(x) = 3^x$ is strictly increasing on R.
10. Show that $f(x) = x^3 - 15x^2 + 75x - 50$ is increasing on R.
11. Show that $f(x) = \left(x - \frac{1}{x}\right)$ is increasing for all $x \in R$, where $x \neq 0$.
12. Show that $f(x) = \left(\frac{3}{x} + 5\right)$ is decreasing for all $x \in R$, where $x \neq 0$.

13. Show that $f(x) = \dfrac{1}{(1+x^2)}$ is increasing for all $x \le 0$.

14. Show that $f(x) = \left(x^3 + \dfrac{1}{x^3}\right)$ is decreasing on $]-1, 1[$. **[CBSE 2009]**

15. Show that $f(x) = \dfrac{x}{\sin x}$ is increasing on $\left]0, \dfrac{\pi}{2}\right[$.

16. Prove that the function $f(x) = \log(1+x) - \dfrac{2x}{(x+2)}$ is increasing for all $x > -1$. **[CBSE 2000C]**

17. Let I be an interval disjoint from $]-1, 1[$. Prove that the function $f(x) = \left(x + \dfrac{1}{x}\right)$ is strictly increasing on I.

18. Show that $f(x) = \dfrac{(x-2)}{(x+1)}$ is increasing for all $x \in R$, except at $x = -1$.

19. Find the intervals on which the function $f(x) = (2x^2 - 3x)$ is (a) strictly increasing (b) strictly decreasing.

20. Find the intervals on which the function $f(x) = 2x^3 - 3x^2 - 36x + 7$ is
(a) strictly increasing (b) strictly decreasing. **[CBSE 2005C]**

21. Find the intervals on which the function $f(x) = 6 - 9x - x^2$ is
(a) strictly increasing (b) strictly decreasing.

Find the intervals on which each of the following functions is (a) increasing (b) decreasing.

22. $f(x) = \left(x^4 - \dfrac{x^3}{3}\right)$

23. $f(x) = x^3 - 12x^2 + 36x + 17$ **[CBSE 2006, '09C]**

24. $f(x) = (x^3 - 6x^2 + 9x + 10)$ **[CBSE 2000, '04C]**

25. $f(x) = (6 + 12x + 3x^2 - 2x^3)$

26. $f(x) = 2x^3 - 24x + 5$

27. $f(x) = (x-1)(x-2)^2$ **[CBSE 2009C]**

28. $f(x) = (x^4 - 4x^3 + 4x^2 + 15)$

29. $f(x) = 2x^3 + 9x^2 + 12x + 15$ **[CBSE 2008C]**

30. $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$ **[CBSE 2012C]**

31. $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is (a) strictly increasing (b) strictly decreasing. **[CBSE 2014]**

32. $f(x) = \dfrac{3}{10}x^4 - \dfrac{4}{5}x^3 - 3x^2 + \dfrac{36}{5}x + 11$ is (a) strictly increasing (b) strictly decreasing. **[CBSE 2014C]**

ANSWERS (EXERCISE 11G)

19. (a) $f(x)$ is strictly increasing on $\left]\frac{3}{4}, \infty\right[$.

(b) $f(x)$ is strictly decreasing on $\left]-\infty, \frac{3}{4}\right[$.

20. (a) $f(x)$ is strictly increasing on $]-\infty, -2[\cup]3, \infty[$.

(b) $f(x)$ is strictly decreasing on $]-2, 3[$.

21. (a) $f(x)$ is strictly increasing on $\left]-\infty, -\frac{9}{2}\right[$.

(b) $f(x)$ is strictly decreasing on $\left]-\frac{9}{2}, \infty\right[$.

22. (a) $f(x)$ is increasing on $\left[\frac{1}{4}, \infty\right[$.

(b) $f(x)$ is decreasing on $\left]-\infty, \frac{1}{4}\right]$.

23. (a) $f(x)$ is increasing on $]-\infty, -2[\cup]6, \infty[$.

(b) $f(x)$ is decreasing on $]2, 6[$.

24. (a) $f(x)$ is increasing on $]-\infty, 1] \cup [3, \infty[$.

(b) $f(x)$ is decreasing on $[-1, 3]$.

25. (a) $f(x)$ is increasing on $[-1, 2]$.

(b) $f(x)$ is decreasing on $]-\infty, -1] \cup [2, \infty[$.

26. (a) $f(x)$ is increasing on $]-\infty, -2] \cup [1, \infty[$.

(b) $f(x)$ is decreasing on $[-2, 2]$.

27. (a) $f(x)$ is increasing on $\left]-\infty, \frac{4}{3}\right] \cup [2, \infty[$.

(b) $f(x)$ is decreasing on $\left[\frac{4}{3}, 2\right]$.

28. (a) $f(x)$ is increasing on $[0, 1] \cup [2, \infty[$.

(b) $f(x)$ is decreasing on $]-\infty, 0] \cup [1, 2]$.

29. (a) $f(x)$ is increasing on $]-\infty, -2] \cup [-1, \infty[$.

(b) $f(x)$ is decreasing on $[-2, -1]$.

30. (a) $f(x)$ is increasing on $[1, 2] \cup [3, \infty[$

(b) $f(x)$ is decreasing on $]-\infty, 1] \cup [2, 3]$

31. (a) Strictly increasing in $]-1, 0[\cup]2, \infty[$.

(b) Strictly decreasing in $]-\infty, -1[\cup]0, 2[$.

32. (a) Strictly increasing in $]-2, 1[\cup]3, \infty[$.

(b) Strictly decreasing in $]-\infty, -2[\cup]1, 3[$.

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 11G)

1. $x_1 > x_2 \Rightarrow 5x_1 > 5x_2$
$\Rightarrow (5x_1 - 2) > (5x_2 - 2)$
$\Rightarrow f(x_1) > f(x_2)$.

2. $x_1 > x_2 \Rightarrow -2x_1 < -2x_2$
$\Rightarrow -2x_1 + 7 < -2x_2 + 7$
$\Rightarrow f(x_1) < f(x_2)$.

5. (a) Let $x_1, x_2 \in [0, \infty[$ such that $x_1 > x_2$. Then,

$x_1 > x_2 \Rightarrow x_1^2 > x_1 x_2$ and $x_1 x_2 > x_2^2$

$\Rightarrow x_1^2 > x_2^2 \Rightarrow f(x_1) > f(x_2)$.

(b) Let $x_1, x_2 \in]-\infty, 0]$ such that $x_1 > x_2$. Then,

$x_1 > x_2 \Rightarrow x_1 x_2 < x_2^2$ $\quad [\because x_2 < 0]$.

Also, $x_1 > x_2 \Rightarrow x_1^2 < x_1 x_2$ $\quad [\because x_1 < 0]$.

$\therefore \; x_1^2 < x_1 x_2 < x_2^2$.

Thus, $x_1 > x_2 \Rightarrow x_1^2 < x_2^2$

$\Rightarrow f(x_1) < f(x_2)$.

(c) Since $f(x) = x^2$ is strictly increasing on $[0, \infty[$ and strictly decreasing on $]-\infty, 0]$, it is neither strictly increasing nor strictly decreasing on the whole real line.

6. (a) Let $x_1, x_2 \in]0, \infty[$ such that $x_1 > x_2$. Then,

$x_1 > x_2 \Rightarrow |x_1| > |x_2|$

$\Rightarrow f(x_1) > f(x_2)$.

$\therefore \; f(x) = |x|$ is strictly increasing on $]0, \infty[$.

(b) Let $x_1, x_2 \in]-\infty, 0[$ such that $x_1 > x_2$. Then,

$x_1 > x_2 \Rightarrow |x_1| < |x_2|$

$\Rightarrow f(x_1) < f(x_2)$.

Thus, $x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$.

$\therefore \; f(x) = |x|$ is strictly decreasing on $]-\infty, 0[$.

7. For all $x_1, x_2 \in]0, \infty[$, we have

$x_1 \geq x_2 \Rightarrow \log_e x_1 \geq \log_e x_2 \Rightarrow f(x_1) \geq f(x_2)$.

8. $f(x) = \log_a x = \dfrac{\log_e x}{\log_e a}$.

Case I Let $x_1, x_2 \in]0, \infty[$ and let $a > 1$.

Then, $x_1 > x_2 \Rightarrow \log x_1 > \log x_2$

$\Rightarrow \dfrac{\log x_1}{\log a} > \dfrac{\log x_2}{\log a}$

$\Rightarrow \log_a x_1 > \log_a x_2$.

$\therefore \; f(x)$ is strictly increasing on $[0, \infty[$ when $a > 1$.

Case II Let $x_1, x_2 \in]0, \infty[$ and let $0 < a < 1$.

Then, $x_1 > x_2 \Rightarrow \log x_1 > \log x_2$

$\Rightarrow \dfrac{\log x_1}{\log a} < \dfrac{\log x_2}{\log a} \quad \left[\because 0 < a < 1 \Rightarrow \log a < 0 \Rightarrow \dfrac{1}{\log a} < 0\right]$

$\Rightarrow (\log_a x_1) < (\log_a x_2)$.

$\therefore$ $f(x)$ is strictly decreasing on $]0, \infty[$ when $0 < a < 1$.

9. $f(x) = 3^x \Rightarrow f'(x) = (3^x \log 3) > 0$ for all $x \in R$.

10. $f(x) = x^3 - 15x^2 + 75x - 50$

$\Rightarrow f'(x) = 3x^2 - 30x + 75 = 3(x^2 - 10x + 25) = 3(x-5)^2 \geq 0$.

11. $f(x) = \left(x - \dfrac{1}{x}\right) \Rightarrow f'(x) = \left(1 + \dfrac{1}{x^2}\right) > 0$ for all $x \in R$, where $x \neq 0$.

12. $f(x) = \left(\dfrac{3}{x} + 5\right) \Rightarrow f'(x) = \dfrac{-3}{x^2} < 0$ for all $x \in R$, where $x \neq 0$.

13. $f'(x) = -(1 + x^2)^{-2} \cdot 2x = \dfrac{-2x}{(1+x^2)^2} \geq 0$ when $x \leq 0$.

14. $f'(x) < 0 \Leftrightarrow \dfrac{3(x^6 - 1)}{x^4} < 0 \Leftrightarrow (x^6 - 1) < 0$

$\Leftrightarrow (x-1)(x+1)(x^4 + x^2 + 1) < 0$

$\Leftrightarrow (x-1)(x+1) < 0 \Leftrightarrow -1 < x < 1$.

15. $f'(x) > 0 \Leftrightarrow \dfrac{\sin x - x \cos x}{\sin^2 x} > 0 \Leftrightarrow (\sin x - x \cos x) > 0$

$\Leftrightarrow \tan x > x$, which is true on $\left]0, \dfrac{\pi}{2}\right[$.

16. Clearly, $\log(1 + x)$ is defined only when $(1 + x) > 0$, i.e., when $x > -1$.

Now, $f(x) = \log(1 + x) - \dfrac{2x}{(x+2)}$

$\Rightarrow f'(x) = \left\{\dfrac{1}{(1+x)} - \dfrac{[(x+2)\cdot 2 - 2x \cdot 1]}{(x+2)^2}\right\} = \dfrac{x^2}{(1+x)(x+2)^2} > 0 \quad [\because (1+x) > 0]$.

17. $f(x) = x + \dfrac{1}{x} \Rightarrow f'(x) = \left(1 - \dfrac{1}{x^2}\right) = \dfrac{(x^2 - 1)}{x^2}$.

Now, $f'(x) > 0 \Leftrightarrow (x^2 - 1) > 0 \Leftrightarrow (x-1)(x+1) > 0$

$\Leftrightarrow [(x-1) > 0 \text{ and } (x+1) > 0]$ or $[(x-1) < 0 \text{ and } (x+1) < 0]$

$\Leftrightarrow [x > 1 \text{ and } x > -1]$ or $[x < 1 \text{ and } x < -1]$

$\Leftrightarrow (x > 1)$ or $(x < -1)$

$\Leftrightarrow x \in]-\infty, -1[$ or $x \in]1, \infty[$.

$\Leftrightarrow x \in]-\infty, -1[\cup]1, \infty[$.